IMO Problems on Functional Equation

1968/5
Let f be a real-valued function defined for all real numbers x such that, for some positive constant a, the equation

$$f(x+a)=\frac{1}{2}+\sqrt{f(x)-(f(x))^2} \text{ holds for all } x.$$

a) Prove that the function f is periodic. (i.e. there exists a positive number b such that $f(x+b)=f(b)$ for all x)

b) For $a=1$, give an example of a non-constant function with the required properties.

1972/5
Let f and g be real-valued functions defined for all real values of x and y, and satisfying the equation $f(x+y)+f(x-y)=2f(x)g(y)$ for all x, y.
Prove that if $f(x)$ is not identically zero, and if $|f(x)|\le 1$ for all x, then $|g(y)|\le 1$ for all y.

1977/6
Let $f(n)$ be a function defined on the set of all positive integers and having all its values in the same set. Prove that if $f(n+1)>f(f(n))$ for each positive integer n, then $f(n)=n$ for each n.

1978/3
The set of all positive integers is the union of two disjoint subsets

$$\{f(1), f(2), \cdots, f(n), \cdots\} \text{ and } \{g(1), g(2), \cdots, g(n), \cdots\},$$

where $f(1)<f(2)<\cdots<f(n)<\cdots$, $g(1)<g(2)<\cdots<g(n)<\cdots$, and $g(n)=f(f(n))+1$ for all $n\ge 1$.
Determine $f(240)$.

1981/6
The function $f(x,y)$ satisfies

(1) $f(0,y)=y+1$,

(2) $f(x+1,0)=f(x,1)$,

(3) $f(x+1,y+1)=f(x,f(x+1,y))$,

for all non-negative integers x, y. Determine $f(4,1981)$.

1982/1
The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n,

$$f(m+n)-f(m)-f(n)= 0 \text{ or } 1$$

$$f(2)=0,\ f(3)>0, \text{ and } f(9999)=3333.$$

Determine $f(1982)$.

1983/1
Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:
(i) $f(xf(y)) = yf(x)$ for all positive x, y;
(ii) $f(x)\to 0$ as $x\to\infty$.

1986/5
Find all functions f, defined on the non-negative real numbers and taking non-negative real values, such that:
(i) $f(xf(y))f(y) = f(x+y)$ for all $x, y \geq 0$,
(ii) $f(2)=0$,
(iii) $f(x)\neq 0$ for $0\leq x<2$.

1987/4
Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n+1987$ for every n.

1988/3
A function f is defined on the positive integers by $f(1)=1$, $f(3)=3$, $f(2n)=f(n)$, $f(4n+1)=2f(2n+1)-f(n)$, $f(4n+3)=3f(2n+1)-2f(n)$ for all positive integers n.
Determine the number of positive integers n, less than or equal to 1988, for which $f(n)=n$.

1990/4
Let Q^+ be the set of positive rational numbers. Construct a function $f: Q^+\to Q^+$ such that

$$f(xf(y)) = \frac{f(x)}{y} \text{ for all } x, y \text{ in } Q^+.$$

1992/2
Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^2 + f(y)) = y+(f(x))^2 \text{ for all } x, y\in\mathbb{R}.$$

1993/5

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$. Determine whether or not there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 2$, $f(f(n)) = f(n) + n$ for all $n \in \mathbb{N}$, and $f(n) < f(n+1)$ for all $n \in \mathbb{N}$.

1994/5

Let S be the set of real number greater than –1. Find all functions $f : S \to S$ satisfying the two conditions

(i) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x and y in S;

(ii) $\dfrac{f(x)}{x}$ is strictly increasing for $-1 < x < 0$ and for $x > 0$.

1996/3

Let $S = \{0, 1, 2, \cdots\}$ be the set of non-negative integers. Find all functions f defined on S and taking their values in S such that

$$f(m + f(n)) = f(f(m)) + f(n) \text{ for all } m, n \text{ in } S.$$

1998/6

Consider all functions f from the set $\mathbb{N}$ of all positive integers into itself satisfying $f(t^2 f(s)) = s(f(t))^2$ for all s and t in $\mathbb{N}$. Determine the least possible value of $f(1998)$.

1999/6

Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.