Generating Functions

In the note, we introduce the concept of a generating function - a powerful tool that is very useful in solving counting problems and recurrence relations, particularly problems involving the selection and arrangement of objects with repetition and with additional constraints.

1. Introduction

Definition

For a sequence $\{a_n\}$, the power series $A(z) = a_0 + a_1z + a_2z^2 + ... = \sum_{n=0}^{\infty} a_n z^n$ is called the generating function of the sequence $\{a_n\}$.

For example, the generating function of the sequence $\{3^n\}$ is

$$A(z) = 1 + 3z + 3^2z^2 + 3^3z^3 + ...$$

and can also be written in the form $A(z) = \frac{1}{1-3z}$, called a **closed form** representation of A(z).

Remark:
When we are using generating functions as tools for manipulating sequences, we would not bother to consider the convergence properties of the generating functions. We rather assume that suitable values of z have been chosen so that the generating function is convergent.

Since many generating functions involve factors of the form $(1-z)^{-n}$, let us introduce a general definition of binomial coefficients.

Definition

For integer k and real number r, the binomial coefficient $\binom{r}{k}$ is defined by

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!} \quad \text{for } k > 0; \quad \binom{r}{0} = 1 \text{ and } \binom{r}{k} = 0 \text{ for } k < 0.$$

By expanding $(1 + z)^r$ into a Taylor series at z = 0, we can prove that

$$(1+z)^r = \sum_{n=0}^{\infty} \binom{r}{n} z^n$$

where the exponent m is any real number. If m is a positive integer, then the series has only a finite number of nonzero terms and is the usual binomial expansion.

In particular, for positive integer m, we have $(1-z)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} (-1)^n z^n$.

Now $\binom{-m}{n} = \frac{(-m)(-m-1)\cdots(-m-n+1)}{n!} = \frac{(-1)^n (m+n-1)(m+n-2)\cdots(m+1)m}{n!} = (-1)^n \binom{m+n-1}{n}$.

Hence we get $(1-z)^{-m} = \sum_{n=0}^{\infty} \binom{m+n-1}{n} z^n$.

2. Generating Function Applied to Combinatorial Problems

Example 2.1
Suppose that n objects are selected from a set with 10 elements belonging to 5 different groups. Group 1 has 2, groups 2 and 3 have 1, and groups 4 and 5 have 3 identical objects. Lets a_n be the number of ways of selecting such n objects. Find a_n.

Solution:
The generating function of $\{a_n\}$ is simply

$$A(z) = (1 + z + z^2)(1 + z)^2(1 + z + z^2 + z^3)^2.$$

Note that the coefficient of z^n in A(z) is the number of ways to make up the term z^n from the five factors. The contribution from the factor $1 + z + z^2$ can be 1, z, or z^2, corresponding to selecting 0, 1,or 2 objects from group 1. The contributions from the other factors similarly correspond to the number of objects selected.
If there is no restriction on the number of times each object can be selected, then we have the following situation.

Example 2.2
Suppose that r objects are selected from n objects with unlimited repetitions. Let a_r denote the number of such selections. Find a_r.

Solution:
The generating function of $\{a_r\}$ is

$$A(z) = (1 + z + z^2 + ...)^n = (1 - z)^{-n}.$$

Since $(1-z)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} z^r$, we have $a_r = \binom{n+r-1}{r}$.

Example 2.3
A set of r balls is selected from an infinite supply of red, blue and white balls with the constraint that the number of red balls selected is an even number or the number of blue balls selected is an odd number. Let a_r denote the number of such selections. Find a_r.

Solution:
The generating function of $\{a_n\}$ is

$$A(z) = (1 + z + z^2 + ...)^2(1 + z^2 + z^4 + ...) + (1 + z + z^2 + ...)^2(z + z^3 + ...) - (1 + z + z^2 + .. .)(1 + z^2 + z^4 + ...)(z + z^3 + ...)$$

Hence $A(z) = \frac{1+z}{(1-z)^2(1-z^2)} - \frac{z}{(1-z)(1-z^2)^2} = \frac{1+z+z^2}{(1-z)^3(1+z)^2}$

Using partial fractions,

$$A(z) = \frac{1}{16}\left[\frac{1}{1-z} + \frac{12}{(1-z)^3} + \frac{1}{1+z} + \frac{2}{(1+z)^2}\right].$$

Thus, $a_r = \frac{1}{16}\left[1 + 12\binom{3+r-1}{r} + (-1)^r + 2(-1)^r\binom{2+r-1}{r}\right] = \frac{1}{16}\left[1 + 12\binom{2+r}{2} + (-1)^r + 2(-1)^r\binom{1+r}{1}\right].$

$$= \frac{1}{16}[1 + 12(r+2)(r+1) + (-1)^r + 2(r+1)(-1)^r].$$

Example 2.4
Find the number of integer solutions of the equation a + b + c + d = 25, where each variable is at least 3 and at most 8.

Solution:
The number of solutions is the coefficient of z^{25} in $A(z) = (z^3 + z^4 + ... + z^8)^4$, and this number is the coefficient of z^{13} in $B(z) = (1 + z + z^2 + ... + z^5)^4$. Now

$$B(z) = \left(\frac{1-z^6}{1-z}\right)^4 = (1-z^6)^4(1-z)^{-4}.$$

Since $(1-z^6)^4 = 1 - \binom{4}{1}z^6 + \binom{4}{2}z^{12} + \cdots$ and $(1-z)^{-4} = 1 + \binom{4}{1}z + \binom{5}{2}z^2 + \cdots$,

$\therefore$ The coefficient of z^{13} in B(z) is equal to $\binom{16}{13} - \binom{4}{1}\binom{10}{7} + \binom{4}{2}\binom{4}{1}$.