The Fifth Hong Kong (China) Mathematical Olympiad

December 21, 2002

Time allowed: 3 hours

Each problem is worth 7 marks

1. Two circles intersect at points A and B. Through the point B a straight line is drawn, intersecting the first circle at K and the second circle at M. A line parallel to AM is tangent to the first circle at Q. The line AQ intersects the second circle again at R.

   (a) Prove that the tangent to the second circle at R is parallel to AK.
   (b) Prove that these two tangents are concurrent with KM.

2. Let $n \geq 3$ be an integer. In a conference there are n mathematicians. Every pair of mathematicians communicate in one of the n official languages of the conference. For any three different official languages, there exist three mathematicians who communicate with each other in these three languages. Determine all n for which this is possible. Justify your claim.

3. If $a \geq b \geq c \geq 0$ and $a+b+c=3$, then prove that $ab^2+bc^2+ca^2 \leq \frac{27}{8}$ and determine the equalities case(s).

4. Let p be an odd prime such that $p \equiv 1 \pmod 4$. Evaluate with reasons, $\sum_{k=1}^{\frac{p-1}{2}} \left\{ \frac{k^2}{p} \right\}$, where $\{x\} = x - [x]$, $[x]$ being the greatest integer not exceeding x.

*****End of Paper*****

Solutions

1.

(a) (Siu Tsz Hang) Let *PR* be the tangent to the second circle at *R*. Join *AB* and *MR*. We have $\angle ARP = \angle MRP + \angle ARM = \angle MAR + \angle ABK = \angle LQA + \angle ABK$ $= \angle QBA + \angle ABK = 180^{\circ} - \angle KAQ$. Thus $AK \| PR$, i.e., the tangent to the second circle at *R* is parallel to *AK*.

(b) Let the two tangents meet at *J*. Note that $\angle BRP = \angle BAR = \angle BQJ$ (tangent property). Thus *BJRQ* is cyclic. On the other hand, by (a) $\angle KBQ = \angle AQJ$. This means *BM* produced will meet *PR* at *J*.

2. (Yu Hok Pun) It is possible if and only if *n* is odd. The problem is equivalent to color the edges of K_n by *n* colors, and such that for any three colors, there exist three vertices such that their edges are of these three colors. Now notice that there are C_3^n triples of colors and there are C_3^n triples of vertices, that means for the edges each triple of vertices they must be colored by a unique triple of colors, and vice versa. In particular for every triangle, the edges must be of different colors. Now fix a color *S*, there exist exactly C_2^{n-1} triangles with one edge of color *S*. Yet an edge of color *S* is connected with $n-2$ vertices (hence they form $n-2$ triangles). Therefore there are $\frac{C_2^{n-1}}{n-2} = \frac{n-1}{2}$ edges of color *S*. Now the condition that $\frac{n-1}{2}$ is an integer cannot be fulfilled if *n* is even.

Assume n is odd, denote the vertices by $1,2,\cdots,n$ and the colors by $S_1,S_2,\cdots,S_n$. Color the edge connecting i and j by the color S_t with $t=i+j\,(\mathrm{mod}\,n)$. Then for every triple of colors S_{t_1},S_{t_2}, and S_{t_3} the system

$$\begin{aligned} i+j &\equiv t_1 \pmod n \\ j+k &\equiv t_2 \pmod n \\ k+i &\equiv t_3 \pmod n \end{aligned}$$

has a unique solution, as the determinant of the coefficient matrix $\begin{vmatrix}1&1&0\\0&1&1\\1&0&1\end{vmatrix}=\begin{vmatrix}1&1&0\\0&1&1\\0&-1&1\end{vmatrix}=\begin{vmatrix}1&1&0\\0&1&1\\0&0&2\end{vmatrix}=2\neq 0,$ since n is odd. Therefore for each triple of colors, there exists a unique triangle with edges of these colors.

3. Let $f(a,b,c)=ab^2+bc^2+ca^2$. Then

$$\begin{aligned} f(a,b,c)+f(a,c,b) &= ab^2+bc^2+ca^2+ac^2+cb^2+ba^2 \\ &= (a+b+c)(ab+bc+ca)-3abc \\ &= 3(ab+bc+ca-abc) \\ &= 3[(1-a)(1-b)(1-c)+(a+b+c)-1] \\ &= 3(1-a)(1-b)(1-c)+6. \end{aligned}$$

Since $a\geq b\geq c\geq 0$, we have $c\leq 1\leq a$ and $b\leq \frac{3}{2}$.

If $b\leq 1$, then $(1-a)(1-b)(1-c)\leq 0$ so that $f(a,b,c)+f(a,c,b)\leq 6$.

If $1<b\leq \frac{3}{2}$, then

$(1-a)(1-b)(1-c)\leq (a-1)(b-1)\leq \left(\frac{(a-1)+(b-1)}{2}\right)^2\leq \left(\frac{3-2}{2}\right)^2=\frac{1}{4}$ so that

$f(a,b,c)+f(a,c,b)\leq 6\frac{3}{4}=\frac{27}{4}$. Note that equality holds if and only if $c=0$ and $a=b=\frac{3}{2}$.

Now $f(a,c,b)-f(a,b,c)=(a-b)(b-c)(a-c)\geq 0$,

which implies $f(a,b,c)\leq f(a,c,b)$. Then $f(a,b,c)\leq \frac{1}{2}\big(f(a,b,c)+f(a,c,b)\big)\leq \frac{27}{8}$.

Equality holds if and only if $c=0$ and $a=b=\frac{3}{2}$.

(Alternate Solution by Yu Hok Pun) Put $x=\frac{c}{3},y=\frac{b-c}{3},z=\frac{a-b}{3}$. Then $x+y+z=\frac{a}{3}$, $x+y=\frac{b}{3}$, $x=\frac{c}{3}$. Add together get $3x+2y+z=1$. Thus $a=\frac{3(x+y+z)}{3x+2y+z}$,

$b=\dfrac{3(x+y)}{3x+2y+z}$, $c=\dfrac{3x}{3x+2y+z}$. By expressing $a, b,$ and c in terms of $x, y,$ and z, get $3x^3+z^3+6x^2y+4xy^2+3x^2z+xz^2+4y^2z+6yz^2+18xyz\ge 0$. As $x,y,z\ge 0$, equality holds if and only if $x=z=0$, and hence $c=0$, $a=b=\dfrac{3}{2}$.

4. First note that $(-k)^2\equiv(p-k)^2\equiv k^2 \pmod p$. Also if $x^2\equiv y^2 \pmod p$, where $1\le x,y\le\dfrac{p-1}{2}$, then $(x-y)(x+y)\equiv 0 \pmod p$. But $1<x+y<p$, hence $x=y$. These imply $1^2, 2^2, \cdots, \left(\dfrac{p-1}{2}\right)^2$ is a reduced residue system modulo p. Now since $p\equiv 1 \pmod 4$, by Euler's criterion, $\left(\dfrac{-1}{p}\right)=(-1)^{\frac{p-1}{2}}=1$, hence -1 is a square. Because $\left(\dfrac{-b}{p}\right)=\left(\dfrac{-1}{p}\right)\left(\dfrac{b}{p}\right)$, thus b is a square if and only if $-b\equiv p-b \pmod p$ is. Therefore the set $\{1^2, 2^2, \cdots, \left(\dfrac{p-1}{2}\right)^2\}$ consists of pairs $\{a_1, p-a_1, a_2, p-a_2, \cdots, a_{\frac{p-1}{4}}, p-a_{\frac{p-1}{4}}\}$ modulo p. As $1<a_i, p-a_i<p$, we have $\{\dfrac{a_i}{p}\}+\{\dfrac{p-a_i}{p}\}=\dfrac{a_i+p-a_i}{p}=1$, thus the sum is $\dfrac{p-1}{4}$.

*****End*****