International Mathematical Olympiad 2003
Hong Kong Team Selection Test 1

31 August 2002

Attempt all 7 Questions
Time Allowed: 3 Hours
All Questions carry Equal Marks

1. Let *a, b and c* be nonnegative real numbers. Prove that

$$\frac{(a+b+c)^2}{3} \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}.$$

2. Let ΔABC be a triangle. M is the mid-point of AC, D is a point on AB. BM and CD meet at O, with $AB = CO$. E is a point on AC such that $DE \parallel BM$. Prove that $AB \perp BC$ if and only if $ADOM$ is a cyclic quadrilateral.

3. Let n be an integer greater than 1. In a school there are $n^2 - n + 2$ clubs and each club has exactly n members. Each pair of clubs has exactly one member in common. Show that there is one student belongs to all of the clubs.

4. Show that there exists a positive integer k such that $k \times 22^n + 1$ is not prime (composite) for every positive integer n.

5. Denote by $\sigma(n)$ the sum of divisors of n. A positive integer n $(n \geq 2)$ is redundant if for any integer k, with $k < n$, we have $\frac{\sigma(k)}{k} < \frac{\sigma(n)}{n}$. (For instance if $n = 1, 2, 3, 4, 5$, then $\frac{\sigma(n)}{n} = 1, \frac{3}{2}, \frac{4}{3}, \frac{7}{4}, \frac{6}{5}$ respectively. Hence 2 and 4 are redundant, 3 and 5 are not.) Show that there exist infinitely many redundant numbers.

6. In a conference, there are 2002 representatives from 100 countries. The number of representatives from each country is at least 1 and at most 45. They are seated in rows with each row consisting of 45 seats. It is required that the representatives from the same country must be seated in the same row. What is the smallest numbers of rows needed to ensure that all representatives can be seated? Justify your answer.

7. Let $n \geq 3$ be an integer and $x_1, x_2, \cdots, x_{n-1}$ be nonnegative integers such that

$$\text{(i) } x_1 + x_2 + \cdots + x_{n-1} = n$$
$$\text{(ii) } x_1 + 2x_2 + \cdots + (n-1)x_{n-1} = 2n - 2.$$

Find the minimum of the sum $\sum_{k=1}^{n-1} kx_k(2n-k)$. Justify your answer.

End of Paper

1. Expanding $(a+b+c)^2$ and applying the AM-GM inequality, we have

$$\begin{aligned}(a+b+c)^2 &= (a^2+bc)+(b^2+ac)+(c^2+ba)+\frac{ab+ca}{2}+\frac{bc+ab}{2}+\frac{ca+bc}{2}\\ &\geq 2a\sqrt{bc}+2b\sqrt{ca}+2c\sqrt{ab}+a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\\ &= 3(a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}).\end{aligned}$$

The desired inequality follows.

2. By scaling we may assume $AB = CO = 1$. Let $DB = x$ and $DO = y$. Considering ΔABM, we see that $EM : AM = BD : AB = x : 1$. Considering ΔCDE, we see that $EM : MC = DO : OC = y : 1$. Since $AM = MC$, we have $x = y$.
Let $\angle DBO = \theta$. Then $\angle DOB = \angle EDA = \angle EDO = \theta$. Consequently $ADOM$ is a cyclic quadrilateral

$$\begin{aligned}&\Leftrightarrow \angle ADO + \angle AMO = 180^\circ\\ &\Leftrightarrow \angle AMO = 180^\circ - 2\theta\\ &\Leftrightarrow \angle MAB = \theta \text{ (consider the angles of } \Delta ABM)\\ &\Leftrightarrow MA = MB = MC\\ &\Leftrightarrow B \text{ lies on the circle with } AC \text{ as diameter}\\ &\Leftrightarrow AB \perp BC.\end{aligned}$$

3. Pick one of the clubs, say C. It has n members and has one member in common with each of the remaining $n^2 - n + 1$ clubs. By the pigeon-hole principle, one of the members of C, say x, is common to at least n other clubs, say $C_1, C_2, \cdots, C_n$. Take another club D. If x does not belong to D, then D has a common member with each of the $n+1$ clubs $C, C_1, C_2, \cdots, C_n$, which must be distinct since each pair of clubs has exactly one member in common. Thus D has at least $n+1$ members, a contradiction. As the choice of D is arbitrary, it follows that x belongs to all the clubs.

4. $22 \equiv -1 \pmod{23}$, thus $22^n \equiv (-1)^n \equiv -1 \pmod{23}$ if n is odd. $22^2 = 484 \equiv -1 \pmod{97}$, hence $22^4 \equiv 1 \pmod{97}$, thus $22^n \equiv 1 \pmod{97}$ for every $n = 4m$. Also $22^2 = 484 \equiv -1 \pmod 5$, thus $22^{4m+2} \equiv -1 \pmod 5$. Let k be a solution of the system:

$$\begin{aligned}x &\equiv 1 \pmod{23}\\ x &\equiv -1 \pmod{97}\\ x &\equiv 1 \pmod 5,\end{aligned}$$

(such a k exists because of the Chinese Remainder Theorem), then

$$\begin{aligned}k \times 22^n + 1 &\equiv 0 \pmod{23} \text{ if } n \text{ is odd}\\ k \times 22^n + 1 &\equiv 0 \pmod{97} \text{ if } n = 4m\\ k \times 22^n + 1 &\equiv 0 \pmod 5 \text{ if } n = 4m+2,\end{aligned}$$

and $k > 1$, thus $k \times 22^n + 1$ is composite for any $n \geq 1$. (A possible k is 4946).

(Notes: Actually the problem is trivial, take $k = 20$, then $20\times(22)^n + 1 \equiv 20\times(1)^n + 1 \equiv 20+1 \equiv 0 \pmod{21}$. The argument does not work only for numbers of the form $k\times 2^n + 1$, which is USAMO-82, no. 4.)

5. $\frac{\sigma(n)}{n} = \sum_{d|n}\frac{d}{n} = \sum_{d|n}\frac{(n/d)}{n} = \sum_{d|n}\frac{1}{d}$. Take $n = m!$. Since $1, 2, \cdots, m$ are divisors of $m!$, we have $\frac{\sigma(n)}{n} = \sum_{d|n}\frac{1}{d} > 1 + \frac{1}{2} + \cdots + \frac{1}{m}$. This implies the sequence $\{\frac{\sigma(n)}{n}\}$ is unbounded. It follows that there exist infinitely many redundant numbers.

6. The answer is 86.
We can use the following algorithm to seat the representatives:
First of all, we arrange the countries according to their number from the most to the least. Then we let the representatives from the first country to be seated in the first row. If the remaining seats of the first row are enough for the second country, then we seat them in the first row. Otherwise they are seated in a new row. This process is repeated until all representatives are seated.
Denote n as the number of rows and P_i as the number of representatives seated in the i^{th} row. We claim that $P_i \ge 23$ for $i < n$. Let the first country seated in the i^{th} and $(i+1)^{\text{th}}$ rows have s and t representatives respectively. From the algorithm, we see that $P_i \ge s \ge t > 45 - P_i$. Therefore $P_i > 45 - P_i$, implying $P_i \ge 23$. Note that the same argument also works when $i = n$ unless all representatives are seated in n rows.
Next we note that there are not more than 86 countries having more than 23 representatives. Assuming the contrary, since $2002 = 23\times 87 + 1$, the only case is that there are 86 countries with 23 representatives and 1 country with 24 representatives. It contradicts that there are 100 countries.
We claim that 86 rows are enough. After seating in 86 rows using the algorithm, if not all representatives are seated, then at most $2002 - 23\times 86 = 24$ representatives are not seated. On the other hand, there are at least $45\times 86 - 2002 = 1868$ vacant seats. Among the 24 unseated representatives, there must not be 23 of them representing the same country. Therefore they can be divided into 2 groups (each group with no more than 22 representatives, and representatives from the same country are in the same group). Note that there are at least 2 rows with not fewer than 22 seats available. Otherwise the number of vacant seats is at most $45 + 21\times 85 = 1830 < 1868$, a contradiction. As a results, they can be seated in the 2 rows.
Here is a case where 86 rows are required. There are 86 countries with 23 representatives, 10 with 2 representatives, and 4 with 1 representative. For the first 86 countries, no 2 countries can be seated in the same row, as $23\times 2 > 45$. Therefore at least 86 rows are needed.

7.

$$\sum_{k=1}^{n-1} k^2 x_k = \sum_{k=1}^{n-1} [(k-1)(k+1)+1] x_k = \sum_{k=1}^{n-1} x_k + \sum_{k=1}^{n-1} (k-1)(k+1) x_k$$

$$\le n + \sum_{k=1}^{n-1} (k-1) n x_k = n + n \sum_{k=1}^{n-1} (k-1) x_k = n + n(2n-2-n) = n^2 - n.$$

Thus $\sum_{k=1}^{n-1} k x_k (2n-k) = 2n(2n-2) - \sum_{k=1}^{n-1} k^2 x_k \ge 2n(2n-2) - n^2 + n = 3n^2 - 3n.$

The inequality becomes equality if $x_1 = n-1, x_2 = x_3 = \cdots = x_{n-2} = 0,$ and $x_{n-1} = 1.$

(The Paper was set by Andy Chan, K. H.Law, T. W. Leung and Kin Li.)