International Mathematical Olympiad

Preliminary Selection Contest 2007 — Hong Kong

Outline of Solutions

Answers:

1. 667	**2.** 691	**3.** 88	**4.** 24
5. $\sqrt{5}-2$	**6.** 4	**7.** 4013	**8.** 366
9. 425	**10.** 4024036	**11.** 29	**12.** $\sqrt{5}-2$
13. 100	**14.** $\frac{85}{22}$	**15.** $\sqrt{3}-1$	**16.** 2550
17. 4681	**18.** 5π	**19.** 2048	**20.** 6
21. 576	**22.** $\frac{23}{177}$	**23.** $\sqrt{10}$	**24.** (Cancelled)
25. $2\sqrt{2}-1$			

Solutions:

1. We don't have to care about zeros as far as sum of digits is concerned, so we simply count the number of occurrences of the non-zero digits. The digit '3' occurs 18 times (10 as tens digits in 30, 31, …, 39 and 8 as unit digits in 3, 13, …, 73). The same is true for the digits '4', '5'. '6' and '7'. A little modification shows that the digits '1' and '2' each occurs 19 times. Finally the digit '8' occurs 11 times (8, 18, 28, …, 78, 80, 81, 82) and the digit '9' occurs 8 times (9, 19, 29, …, 79). Hence the answer is

$$(3+4+5+6+7)\times 18+(1+2)\times 19+8\times 11+9\times 8=667\ .$$

2. Since $2007 \equiv 7$ (mod 1000), we need n for which $7n \equiv 837$ (mod 1000). Noting that $7 \times 143 = 1001 \equiv 1$ (mod 1000), we multiply both sides of $7n \equiv 837$ (mod 1000) by 143 to get $7n(143) \equiv 837(143)$ (mod 1000), or $n \equiv 691$ (mod 1000). Hence the answer is 691.

Alternative Solution. To find the smallest n for which $7n \equiv 837$ (mod 1000), we look for the smallest multiple of 7 in the sequence 837, 1837, 2837, … It turns out that 4837 is the first term divisible by 7, and hence the answer is $4837 \div 7 = 691$.

3. Note that N is the mid-point of BC, so $NB = NC = 30$. By Pythagoras' Theorem, we have

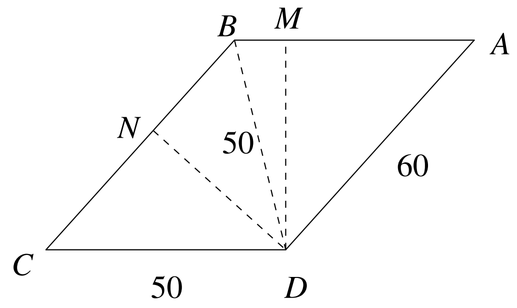
$$DN = \sqrt{50^2 - 30^2} = 40.$$

Computing the area of $ABCD$ in two ways, we get

$$60 \times 40 = 50 \times DM,$$

which gives $DM = 48$. It follows that

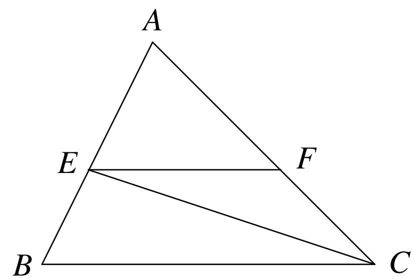
$$DM + DN = 48 + 40 = 88.$$



4. We resort to the formula 'Distance = Speed × Time'. Here 'distance' refers to the total length of the tunnel and the truck. When the speed is reduced by 20% and the time taken is increased by half, the distance must be increased by $(1-20\%) \times (1+50\%) - 1 = 20\%$. Hence, if we let the answer be x m, then we have $\frac{6+x}{12+x} = \frac{5}{6}$, which gives $x = 24$.

5. Using EF and BC as bases, let the heights of ΔAEF and ΔEBC to be k times and $1-k$ times that of ΔABC respectively. Let $[XYZ]$ denote the area of ΔXYZ. Since $\Delta AEF \sim \Delta ABC$, we have $[AEF] = k^2$; since ΔEBC shares the same base as ΔABC, we have $[EBC] = 1-k$. Hence we get $k^2 = 1-k$, and thus $k = \frac{-1+\sqrt{5}}{2}$ as $k > 0$. It follows that

$$[EFC] = 1 - k^2 - (1-k) = \sqrt{5} - 2.$$



6. Note that for positive real number x, we have $[x] > x-1$ and $x+\frac{1}{x} \geq 2$. Hence

$$\begin{aligned}\left[\frac{p+q}{r}\right]+\left[\frac{q+r}{p}\right]+\left[\frac{r+p}{q}\right] &> \left(\frac{p+q}{r}-1\right)+\left(\frac{q+r}{p}-1\right)+\left(\frac{r+p}{q}-1\right)\\ &= \left(\frac{p}{q}+\frac{q}{p}\right)+\left(\frac{q}{r}+\frac{r}{q}\right)+\left(\frac{r}{p}+\frac{p}{r}\right)-3\\ &\geq 2+2+2-3\\ &= 3\end{aligned}$$

Since $\left[\frac{p+q}{r}\right]+\left[\frac{q+r}{p}\right]+\left[\frac{r+p}{q}\right]$ is an integer, it must be at least 4. Such minimum is attainable, for example when $p=6$, $q=8$ and $r=9$.

7. Note that $n=10^{2008}-1$. Hence

$$n^3=(10^{2008}-1)^3=10^{6014}-3\times 10^{4016}+3\times 10^{2008}-1=\underbrace{999\ldots 99}_{2007\text{ digits}}7\underbrace{000\ldots 00}_{2007\text{ digits}}2\underbrace{999\ldots 99}_{2008\text{ digits}}$$

and so the answer is $2007+2008=4015$.

8. Let $x+2y=5a$ and $x+y=3b$. Solving for x, y in terms of a and b, we have $x=6b-5a$ and $y=5a-3b$. Also, $2x+y=9b-5a$ and $7x+5y=27b-10a$. Hence the problem becomes minimising $27b-10a$ over non-negative integers a, b satisfying $6b \geq 5a \geq 3b$ and $9b-5a \geq 99$.

Clearly standard linear programming techniques will solve the problem. We present a purely algebraic solution below. We first establish lower bounds on a and b: $9b \geq 5a+99 \geq 3b+99$ gives $b \geq 17$, while $5a \geq 3b \geq 3(17)$ gives $a \geq 11$. Next $9b \geq 5a+99 \geq 5(11)+99$ gives $b \geq 18$. We consider two cases.

- If $b=18$, then we have $5a \leq 9b-99=9(18)-99=63$, which gives $a \leq 12$ and hence $27b-10a \geq 27(18)-10(12)=366$. We check that $(a,b)=(12,18)$ satisfies all conditions with $27b-10a=366$.
- If $b>18$, then $27b-10a=9b+2(9b-5a) \geq 9(19)+2(99)=369$.

Combining the two cases, we see that the answer is 366.

9. We have $\overline{abc}+2017=\overline{abc}+\overline{acb}+\overline{bac}+\overline{bca}+\overline{cab}+\overline{cba}=222(a+b+c)$. Since $\overline{abc}$ is between 111 and 999, $222(a+b+c)$ is between 2128 and 3016, so that $a+b+c$ must be 10, 11, 12 or 13.

   If $a+b+c=13$, then $\overline{abc}=222\times13-2017=869$, leading to the contradiction $13=8+6+9$. Similar contradictions arise if $a+b+c$ equals 10 or 12. Finally, $a+b+c=11$, then $\overline{abc}=222\times11-2017=425$ and we check that $4+2+5$ is indeed equal to 11. Hence the answer is 425.

10. For a positive integer *n*, let $p(n)$ denote its greatest odd divisor. Then $n=2^k p(n)$ for some nonnegative integer *k*. Hence if $p(r)=p(s)$ for $r\neq s$, then one of *r* and *s* is at least twice the other. Because no number from 2007, 2008, …, 4012 is at least twice another, $p(2007)$, $p(2008)$, …, $p(4012)$ are 2006 distinct odd positive integers. Note further that each of them must be one of the 2006 odd numbers 1, 3, 5, …, 4011. It follows that they are precisely 1, 3, 5, …, 4011 up to permutation, and so the answer is

$$1+3+5+\cdots+4011=2006^2=4024036.$$

11. We have $56=A_1A_{11}=A_1A_2+A_2A_5+A_5A_8+A_8A_{11}\geq A_1A_2+17+17+17$, so that $A_1A_2\leq5$. On the other hand we have $A_1A_2=A_1A_4-A_2A_4\geq17-12=5$, so we must have $A_1A_2=5$. Similarly $A_{10}A_{11}=5$.

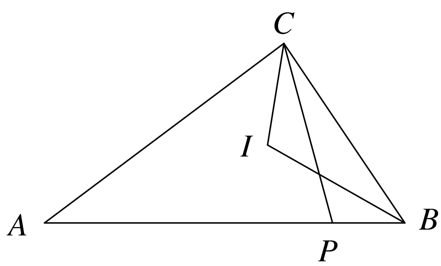
    Next we have $A_2A_7=A_1A_4+A_4A_7-A_1A_2\geq17+17-5=29$ on one hand, and on the other hand we have $A_2A_7=A_1A_{11}-A_1A_2-A_7A_{10}-A_{10}A_{11}\leq56-5-17-5=29$. It follows that $A_2A_7=29$.

    Remark. The scenario in the question is indeed possible. One example is $A_iA_{i+1}=5$, 7, 5, 5, 7, 5, 5, 7, 5, 5 for *i* = 1, 2, …, 10.

12. Let *A*, *B*, *C* denote the corresponding interior angles of ΔABC. Noting that *B*, *P*, *I*, *C* are concyclic, we have

$$\angle APC=180°-\angle CPB=180°-\angle CIB=\frac{B}{2}+\frac{C}{2}$$

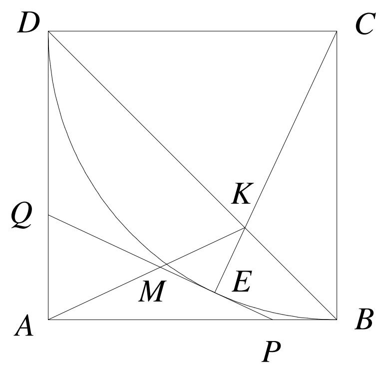
$$\angle ACP=180°-A-\angle APC=\frac{B}{2}+\frac{C}{2}$$



It follows that $AP=AC=2$ and hence $BP=\sqrt{5}-2$.

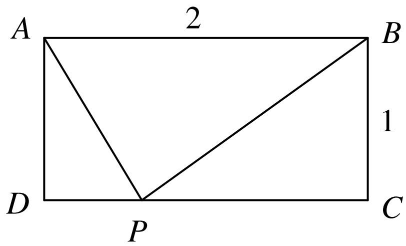
13. We have $3M\geq(x_1+x_2)+(x_2+x_3)+(x_4+x_5)=300+x_2\geq300$, so $M\geq100$. This minimum is attainable when $x_1=x_3=x_5=100$ and $x_2=x_4=0$. Hence the answer is 100.

14. By tangent properties, we have $PE = PB = 2$ and we may let $QD = QE = x$. Then $QA = 9 - x$ and $PQ = x + 2$. Applying Pythagoras' Theorem in ΔAPQ, we have $(9-x)^2 + 7^2 = (x+2)^2$, which gives $x = \frac{63}{11}$.



Next observe that $CEPB$ is a cyclic quadrilateral, so we have $\angle MPA = \angle KCB = \angle KAB$, which means $MP = MA$. Likewise we have $MQ = MA$ and hence $AM = \frac{1}{2}PQ = \frac{1}{2}\left(\frac{63}{11} + 2\right) = \frac{85}{22}$.

15. Note that the requirement of the question is satisfied if and only if both AP and BP are smaller than 2, or equivalently, both DP and CP are smaller than $\sqrt{3}$. Hence if DC is part of the number line with D representing 0 and C representing 2, then P must be between $2-\sqrt{3}$ and $\sqrt{3}$ in order to satisfy the condition, and the probability for this to happen is



$$\frac{\sqrt{3} - (2 - \sqrt{3})}{2} = \sqrt{3} - 1.$$

16. Let d be the H.C.F. of a and c. Then we may write $a = dx$ and $c = dy$ where x, y are integers with an H.C.F. of 1. Then we have

$$b = \frac{ac}{a+c} = \frac{dxy}{x+y} \text{ and } d(x-y) = 101.$$

Since x and y are relatively prime, the first equation shows that $x + y$ must divide d. Hence $d > 1$ and we know from the second equation that $d = 101$ and $x - y = 1$. Since 101 is prime, we must then have $x + y = 101$ and thus $x = 51$, $y = 50$ and $b = \frac{101 \times 50 \times 51}{101} = 2550$.

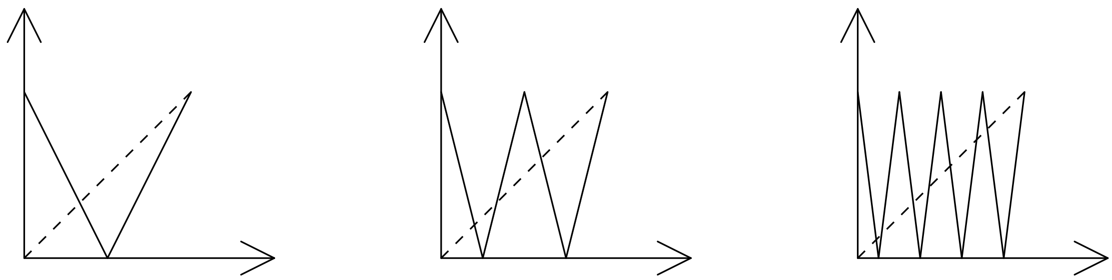
17. The sum of the numbers on all balls is $a + b = 2^0 + 2^1 + \cdots + 2^{14} = 2^{15} - 1 = 32767$. Since both a and b are divisible by d, so is $a + b = 32767$. It is easy to see that d cannot be equal to 32767. Since the prime factorisation of 32767 is $7 \times 31 \times 151$, the next largest possible candidate for d is $31 \times 151 = 4681$. This value of d is attainable; to see this, note that the binary representation of 4681 is 1001001001001, so that if balls numbered 2^k are red if k is divisible by 3 and blue if otherwise, then we will have $a = 4681$, $b = 4681 \times 6$ and $d = 4681$.

18. Being a quadratic equation in $\tan x$ with discriminant $8^2 - 4(3)(3) > 0$, we see that there are two possible values of $\tan x$ (say $\tan x_1$ and $\tan x_2$) and hence four possible values of x in the range $0 < x < 2\pi$. Since $\tan x_1 + \tan x_2 = -\frac{8}{3}$ and $\tan x_1 \tan x_2 = \frac{3}{3} = 1$, both $\tan x_1$ and $\tan x_2$ are negative, so we have two possible values of x in the second quadrant and two in the fourth quadrant.

Consider x_1, x_2 in the second quadrant satisfying the equation. From $\tan x_1 \tan x_2 = 1$, we get $\tan x_2 = \cot x_1 = \tan\left(\frac{3\pi}{2} - x_1\right)$. As $\frac{3\pi}{2} - x_1$ is also in the second quadrant, we must have $x_2 = \frac{3\pi}{2} - x_1$. On the other hand, it is easy to see that the two possible values of x in the fourth quadrant are simply $x_1 + \pi$ and $x_2 + \pi$, as $\tan(x + \pi) = \tan x$ for all x (except at points where $\tan x$ is undefined). It follows that the answer is

$$x_1 + \left(\frac{3\pi}{2} - x_1\right) + (x_1 + \pi) + \left(\frac{3\pi}{2} - x_1 + \pi\right) = 5\pi .$$

19. In an iterative manner, we can work out the graphs of f_0, f_1 and f_2 (together with the graph of $y = x$ in dotted line) in the range $0 \le x \le 1$, which explain everything:



Based on the pattern, it is easy to see (and work out an inductive proof) that the graph of f_{10} consist of $2^{10} = 1024$ copies of 'V', and hence $1024 \times 2 = 2048$ intersections with the line $y = x$. This gives 2048 as the answer.

20. Rewrite the equation as $x\sqrt{y} + y\sqrt{x} + \sqrt{2007xy} = \sqrt{2007x} + \sqrt{2007y} + 2007$, which becomes $\sqrt{xy}\left(\sqrt{x} + \sqrt{y} + \sqrt{2007}\right) = \sqrt{2007}\left(\sqrt{x} + \sqrt{y} + \sqrt{2007}\right)$ upon factorisation. Since the common factor $\sqrt{x} + \sqrt{y} + \sqrt{2007}$ is positive, we must have $xy = 2007 = 3^2 \times 223$. As 2007 has $(2+1)(1+1) = 6$ positive factors, the answer is 6.

21. Let n be such an integer. Note that n is divisible by 4950 if and only if it is divisible by each of 50, 9 and 11. As no two digits of n are the same, the last two digits of n must be 50. Note also that n misses exactly one digit from 0 to 9 with every other digit occurring exactly once. As n is divisible by 9 and the unit digit of n is 0, it is easy to check that the digit missing must be 9, and hence the first 7 digits of n must be some permutation of 1, 2, 3, 4, 6, 7, 8.

Let $n = \overline{ABCDEFG50}$. Since n is divisible by 11, we have $A+C+E+G \equiv B+D+F+5$ (mod 11). Also, the sum of A to G is $1+2+3+4+6+7+8=31$. It is easy to check that the only possibility is $A+C+E+G=18$ and $B+D+F=13$.

Now there are 4 ways to choose 3 digits from {1, 2, 3, 4, 6, 7, 8} to make up a sum of 13, namely, {8, 4, 1}, {8, 3, 2}, {7, 4, 2} and {6, 4, 3}. Each of these gives rise to 3! choices for the ordered triple (B, D, F), and another 4! choices for the ordered quadruple (A, C, E, G). It follows that the answer is $4\times 3!\times 4! = 576$.

22. Note that S contains $3\times 4\times 5 = 60$ elements. Divide the points of S into 8 categories according to the parity of each coordinate. For instance, 'EEO' refers to the points of S whose x- and y-coordinates are even and whose z-coordinate is odd (and similarly for the other combinations of E and O). There are $2\times 2\times 2 = 8$ points in the 'EEO' category. In a similar manner we can work out the size of the other categories:

Category	EEE	EEO	EOE	EOO	OEE	OEO	OOE	OOO
Size	12	8	12	8	6	4	6	4

Note further that two points form a favourable outcome if and only if they are from the same category. It follows that the required probability is

$$\frac{C_2^{12}+C_2^8+C_2^{12}+C_2^8+C_2^6+C_2^4+C_2^6+C_2^4}{C_2^{60}} = \frac{23}{177}.$$

23. Let A, B, P be the points (0, −1), (1, 2) and $\left(x, \frac{1}{x}\right)$ respectively. Then

$$\begin{aligned}\frac{\sqrt{x^4+x^2+2x+1}+\sqrt{x^4-2x^3+5x^2-4x+1}}{x} &= \sqrt{x^2+1+\frac{2}{x}+\frac{1}{x^2}}+\sqrt{x^2-2x+5-\frac{4}{x}+\frac{1}{x^2}} \\ &= \sqrt{x^2+\left(\frac{1}{x}+1\right)^2}+\sqrt{(x-1)^2+\left(\frac{1}{x}-2\right)^2} \\ &= PA+PB \\ &\geq AB \\ &= \sqrt{(0-1)^2+(-1-2)^2} \\ &= \sqrt{10}\end{aligned}$$

Equality is possible if A, P, B are collinear, i.e. P is the intersection of the curve $xy=1$ with the straight line AB in the first quadrant. One can find by simple computation that $x=\frac{1+\sqrt{13}}{6}$ in this case. Hence the answer is $\sqrt{10}$.

24. (This question was cancelled in the live paper. We present below a solution to find the sum of all possible values of S *counting multiplicities*, i.e. if a possible value of S is attained in two different situations, we shall sum that value twice.)

Let's also take into consideration the case where the number of red balls is a positive odd number, and in this case let T be the product of the numbers on all red balls. Let G be the sum of all possible values of S (counting multiplicities) and H be the sum of all possible values of T (counting multiplicities). Note that

$$G+H=\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{1000}\right)-1=\frac{3}{2}\times\frac{4}{3}\times\cdots\times\frac{1001}{1000}-1=\frac{1001}{2}-1=\frac{999}{2}$$

since each possible value of S or T arises from a choice of whether each ball is red or blue, and the term −1 in the end eliminates the case where all balls are blue which is not allowed by the question. In a similar manner, we have

$$G-H=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{1000}\right)-1=\frac{1}{2}\times\frac{2}{3}\times\cdots\times\frac{999}{1000}-1=\frac{1}{1000}-1=-\frac{999}{1000}$$

since each possible value of T (corresponding to a choice of an odd number of red balls) is evaluated −1 time in the above product while each possible value of S is evaluated +1 time. It follows that

$$G=\frac{1}{2}\left(\frac{999}{2}-\frac{999}{1000}\right)=\frac{1}{2}\left(\frac{499500-999}{1000}\right)=\frac{498501}{2000}.$$

25. Let $a=\sin x$ and $b=\cos x$. Then $a^2+b^2=1$ and we want to minimise

$$\left|a+b+\frac{a}{b}+\frac{b}{a}+\frac{1}{b}+\frac{1}{a}\right|=\left|a+b+\frac{1+a+b}{ab}\right|.$$

If we set $c=a+b$, then $ab=\frac{(a+b)^2-(a^2+b^2)}{2}=\frac{c^2-1}{2}$ and so the above quantity becomes

$$\left|c+\frac{2(1+c)}{c^2-1}\right|=\left|c+\frac{2}{c-1}\right|=\left|(c-1)+\frac{2}{c-1}+1\right|.$$

Let $f(c)$ denote this quantity. As $c=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$, c may take values between $-\sqrt{2}$ and $\sqrt{2}$. On the other hand, for positive real number r we have $r+\frac{2}{r}=\left(\sqrt{r}-\sqrt{\frac{2}{r}}\right)^2+2\sqrt{2}\geq 2\sqrt{2}$, with equality when $\sqrt{r}=\sqrt{\frac{2}{r}}$, i.e. $r=\sqrt{2}$. Hence

- if $c>1$, then $f(c)\geq 2\sqrt{2}+1$;
- if $c<1$, then $f(c)=\left|1-\left[(1-c)+\frac{2}{1-c}\right]\right|\geq\left|1-2\sqrt{2}\right|=2\sqrt{2}-1$.

Equality in the second case holds if $1-c=\sqrt{2}$, i.e. if $c=1-\sqrt{2}$, which is possible since $-\sqrt{2}<1-\sqrt{2}<\sqrt{2}$. It follows that the answer is $2\sqrt{2}-1$.