International Mathematical Olympiad 2003
Hong Kong Preliminary Selection Contest

May 26, 2002

Answer ALL questions | Time allowed: 3 hours

Put your answers on the answer sheet

The use of calculator is NOT allowed

1. (1 mark) Let n be a positive integer such that no matter how 10^n is expressed as the product of two positive integers, at least one of these two integers contains the digit 0. Find the smallest possible value of n.
2. (1 mark) A clock has an hour hand of length 3 and a minute hand of length 4. From 1:00 am to 1:00 pm of the same day, find the number of occurrences when the distance between the tips of the two hands is an integer.
3. (1 mark) Find the sum of all integers from 1 to 1000 which contain at least one "7" in their digits.
4. (1 mark) A multiple choice test consists of 100 questions. If a student answers a question correctly, he will get 4 marks; if he answers a question wrongly, he will get –1 mark. He will get 0 mark for an unanswered question. Determine the number of different total marks of the test. (A total mark can be negative.)
5. (1 mark) A positive integer is said to be a "palindrome" if it reads the same from left to right as from right to left. For example 2002 is a palindrome. Find the sum of all 4-digit palindromes.
6. (1 mark) Points A and B lie on a plane. A straight line passing through A will divide the plane into 2 regions. A further straight line through B will altogether divide the plane into 4 regions, and so on. If 1002 and 1000 straight lines are drawn passing through A and B respectively, what is the maximum number of regions formed?
7. (1 mark) In ΔABC, X, Y are points on BC such that $BX = XY = YC$, M, N are points on AC such that $AM = MN = NC$. BM and BN intersect AY at S and R respectively. If the area of ΔABC is 1, find the area of $SMNR$.
8. (1 mark) Given that $0.3010 < \log 2 < 0.3011$ and $0.4771 < \log 3 < 0.4772$. Find the leftmost digit of 12^{37}.
9. (1 mark) Let x_1, y_1, x_2, y_2 be real numbers satisfying the equations $x_1^2 + 5x_2^2 = 10$, $x_2 y_1 - x_1 y_2 = 5$ and $x_1 y_1 + 5x_2 y_2 = \sqrt{105}$. Find the value of $y_1^2 + 5y_2^2$.
10. (1 mark) How many positive integers less than 500 have exactly 15 positive integer factors?
11. (1 mark) Find the 2002^{nd} positive integer that is not the difference of two square integers.
12. (1 mark) In trapezium $ABCD$, $BC \perp AB, BC \perp CD$ and $AC \perp BD$. Given $AB = \sqrt{11}$ and $AD = \sqrt{1001}$. Find BC.
13. (2 marks) Let $ABCD$ be a square of side 5, E a point on BC such that $BE = 3, EC = 2$. Let P be a variable point on the diagonal BD. Determine the length of PB if $PE + PC$ is smallest.

14. (2 marks) In ΔABC, $\angle ACB = 3\angle BAC$, $BC = 5, AB = 11$. Find AC.
15. (2 marks) In ΔABC, D, E and F are respectively the midpoints of AB, BC and CA. Furthermore $AB = 10, CD = 9, CD \perp AE$. Find BF.
16. (2 marks) Each face and each vertex of a regular tetrahedron is coloured red or blue. How many different ways of colouring are there? (Two tetrahedrons are said to have the same colouring if we can rotate them suitably so that corresponding faces and vertices are of the same colour.
17. (2 marks) Let $a_0 = 2$ and for $n \geq 1$, $a_n = \dfrac{\sqrt{3}\, a_{n-1} + 1}{\sqrt{3} - a_{n-1}}$. Find the value of a_{2002} in the form $p + q\sqrt{3}$ where p and q are rational numbers.
18. (2 marks) Let $A_1 A_2 \cdots A_{2002}$ be a regular 2002-sided polygon. Each vertex A_i is associated with a positive integer a_i such that the following condition is satisfied: If $j_1, j_2, \cdots, j_k$ are positive integers such that $k < 500$ and $A_{j_1} A_{j_2} \cdots A_{j_k}$ is a regular $k-$sided polygon, then the values of $a_{j_1}, a_{j_2}, \cdots, a_{j_k}$ are all different. Find the smallest possible value of $a_1 + a_2 + \cdots + a_{2002}$.
19. (3 marks) There are 5 points on the plane. The following steps are used to construct lines. In step 1, connect all possible pairs of the points; it is found that no two lines are parallel, nor any two lines perpendicular to each other, also no three lines are concurrent. In step 2, perpendicular lines are drawn from each of the five given points to straight lines connecting any two of the other four points. What is the maximum number of points of intersection formed by *the lines drawn in step 2*, including the 5 given points?
20. (3 marks) A rectangular piece of paper has integer side lengths. The paper is folded so that a pair of diagonally opposite vertices coincide, and it is found that the crease is of length 65. Find a possible value of the perimeter of the paper.

End of Paper