International Mathematical Olympiad 2001
Hong Kong Preliminary Selection Contest
(Sponsored by the Quality Education Fund)
Solutions

1. (1 mark) Find the sum of all real x satisfying $(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3$.
   Solution: Let $a = 2^x - 4$, and $b = 4^x - 2$, then the equation becomes
   $a^3 + b^3 = (a + b)^3$, giving $3ab(a + b) = 0$
   Either a=0, or b = 0, or a + b = 0
   i.e., $2^x - 4 = 0$, or $4^x - 2 = 0$, or $4^x + 2^x - 6 = (2^x + 3)(2^x - 2) = 0$
   Get x = 2, 1/2 or 1. The sum is 7/2 .

2. (1 mark) In how many ways can 30! be expressed as the product of two integers p and q such that $0 < \frac{p}{q} < 1$ and p and q are relatively prime.
   Solution: There are 10 prime factors of 30!, namely 2, 3, 5, 7, 11, 13, 17, 19, 23, 29.
   Actually, $30! = 2^\alpha \times 3^\beta \times 5^7 \times 7^4 \times 11^2 \times 13 \times 17 \times 19 \times 23 \times 29$.
   Each of these (e.g. 2^α and 3^β), can only appear in one piece in either p or q, this gives 1024 choices. Half of them (so that p < q) will be 512 choices.
   Considering negative integers as well, there are 1024 choices.

3. (1 mark) Find the coefficient of x^{17} in the expansion of $(1 + x^5 + x^7)^{20}$.
   Solution: x^{17} can only be obtained by multiplying two x^5s and one x^7. There are 20 ways to get x^7 and $^{19}C_2 = 171$ ways to get two x^5s in the remaining 19 factors. So the answer is 20 X 171 = 3420.

4. (1 mark) If [x] represents the greatest integer less than or equal to x, find the sum of
   $$\left[\frac{1\times 1999}{2001}\right] + \left[\frac{2\times 1999}{2001}\right] + \left[\frac{3\times 1999}{2001}\right] + \cdots + \left[\frac{2000\times 1999}{2001}\right].$$
   Solution: Note that if n is an integer and x is not an integer, then
   [n + x] = n + [x] and [-x] = -1 – [x]
   $$\text{Hence } \left[\frac{2000 \times 1999}{2001}\right] = \left[1999 - \frac{1999}{2001}\right] = 1999 - 1 - \left[\frac{1999}{2001}\right]$$
   $$\text{Therefore } \left[\frac{1 \times 1999}{2001}\right] + \left[\frac{2000 \times 1999}{2001}\right] = 1999 - 1 = 1998$$
   $$\text{Similarly } \left[\frac{2 \times 1999}{2001}\right] + \left[\frac{1999 \times 1999}{2001}\right] = 1999 - 1 = 1998$$
   $$\left[\frac{3 \times 1999}{2001}\right] + \left[\frac{1998 \times 1999}{2001}\right] = 1999 - 1 = 1998$$

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$$\left[\frac{1000 \times 1999}{2001}\right] + \left[\frac{1001 \times 1999}{2001}\right] = 1999 - 1 = 1998$$

Summing up

$$\left[\frac{1 \times 1999}{2001}\right] + \left[\frac{2 \times 1999}{2001}\right] + \left[\frac{3 \times 1999}{2001}\right] + \ldots + \left[\frac{2000 \times 1999}{2001}\right] = 1{,}998{,}000$$

5. (1 mark) If x, y are nonzero numbers satisfying $x^2 + xy + y^2 = 0$. Find the value of

$$\left(\frac{x}{x+y}\right)^{2001} + \left(\frac{y}{x+y}\right)^{2001}.$$

Solution: Let $t = x/(x + y)$.

Note that $[x/(x + y)]\,[y/(x + y)] = xy/(x^2 + 2xy + y^2) = xy/xy = 1$

Therefore $y/(x + y) = 1/t$, and $t + 1/t = 1$.

$t^2 - t + 1 = 0$, and $(t^3 + 1) = 0$.

Hence $t^3 = -1$

$$\left(\frac{x}{x+y}\right)^{2001} + \left(\frac{y}{x+y}\right)^{2001} . = (t^3)^{667} + (1/t^3)^{667} = -2 .$$

6. (1 mark) For how many real numbers a do the quadratic equations $x^2 + ax + 8a = 0$ have only integral roots?

Solution: Let m, n be the integral roots of the equation, with $m \leq n$.

Then $m + n = -a$ and $mn = 8a$.

Hence $8(m + n) = -mn$, and $mn + 8m + 8n = 0$, $(m+8)(n+8)= 64$.

$64 = 1 \times 64 = 2 \times 32 = 4 \times 16 = 8 \times 8 = -64 \times -1 = -32 \times -2 = -16 \times -4 = -8 \times -8$

giving the solutions (-7, 56), (-6, 24), (-4, 8), (0, 0), (-72, -9), (-40, -10), (-24, -12) and (-16, -16).

Hence a can have 8 different values, namely –49, -18, -4, 0, 81, 50, 36 and 32.

7. (1 mark) Suppose tan α and tan β are the roots of $x^2 + \pi x + \sqrt{2} = 0$. Evaluate

$\sin^2(\alpha + \beta) + \pi \sin(\alpha + \beta)\cos(\alpha + \beta) + \sqrt{2}\cos^2(\alpha + \beta)$.

Solution: Using the formulae for sum of rots and product of roots,

$\tan\alpha + \tan\beta = -\pi$, $(\tan\alpha)(\tan\beta) = \sqrt{2}$

Therefore $\tan(\alpha + \beta) = -\pi/(1 - \sqrt{2})$

Now $\sin^2(\alpha + \beta) + \pi \sin(\alpha + \beta)\cos(\alpha + \beta) + \sqrt{2}\cos^2(\alpha + \beta)$.

$= \cos^2(\alpha + \beta)[\tan^2(\alpha + \beta) + \pi \tan(\alpha + \beta) + \sqrt{2}]$

$= [\tan^2(\alpha + \beta) + \pi \tan(\alpha + \beta) + \sqrt{2}]/[1 + \tan^2(\alpha + \beta)]$

$$= \frac{(1-\sqrt{2})^2}{\pi^2+(1-\sqrt{2})^2} \times \left(\frac{\pi^2}{(1-\sqrt{2})^2} - \frac{\pi^2}{(1-\sqrt{2})^2} + \sqrt{2} \right)$$
$$= \sqrt{2}$$

8. (1 mark) 2000 lamps are controlled by 2000 switches, numbered 1, 2, 3, …, 2000. A click on each switch will either turn the lamp on or off. In the beginning, all the lamps are off. On the first day, all the switches are clicked once. On the second day, all the switches numbered 2 or a multiple of 2 are clicked once. Similarly on the n^{th} day, all the switches numbered n or a multiple of n are clicked once, and so on. How many lamps will be on after the operation on the 2000^{th} day?

Solution: After the 2000^{th} operation, only those lamps with numbers which have an odd number of integral factors will be left open.
This is equal to the number of perfect squares less than 2000.
Since $44^2 = 1936$, and $45^2 = 2025$,
Therefore the number of lamps which are on = 44.

9. (1 mark) Point B is in the exterior of the regular n-sided polygon $A_1A_2 \ldots A_n$ and A_1A_2B is an equilateral triangle. Find the largest value of n such that A_n, A_1 and B are consecutive vertices of a regular polygon.

Solution: Let m be the number of sides of regular polygon with An, A1 and B as consecutive vertices.
The degree measure of the interior angles of the three polygons are
$180 - 360/n$, 60 and $180 - 360/m$.
Hence $180 - 360/n + 60 + 180 - 360/m = 360$
Giving $n = 6 + 36/(m - 6)$
n is largest when $m = 7$, and $n = 42$.

10. (1 mark) There are three parallel lines L_1, L_2 and L_3 on the plane, with L_2 in between. The distance between L_1 and L_2 is 4, and the distance between L_2 and L_3 is 3. A, B and C are points on L_1, L_2 and L_3 respectively, such that ΔABC is an equilateral triangle. Find the area of the triangle.

Solution: Let the circumcircle of ΔABC cut L_2 at the point P.
Note that $\angle APT = \angle BPT = 60°$
$AP = 4/\sin 60° = 8/\sqrt{3}$, $BP = 3/\sin 60° = 6/\sqrt{3}$
$AB = \sqrt{[64/3 + 36/3 - 2(8/\sqrt{3})(6/\sqrt{3}) \cos 120°]} = 148/3$
Area of $\Delta ABC = (\sqrt{3}/4)(148/3) = 37\sqrt{3}/3$.

11. (1 mark) A circle is inscribed in ΔABC. D, E are points on AB and AC respectively, such that DE is parallel to BC and is tangent to the circle. If the perimeter of ΔABC is p, find the maximum length of DE.

Solution: Let BC = a, and DE = x
Using tangent property, the perimeter of $\Delta ADE = p - 2a$
Using property of similar triangles, $x/a = (p - 2a)/p$
$x = (ap - 2a^2)/p = 2[(p/4)^2 - (a - p/4)^2]/p$
x is maximum when $a = p/4$, and $x_{max} = p/8$.

12. (1 mark) In ΔABC, BC = 5, AC = 12, AB = 13. D, E are points on AB and AC respectively such that DE divides ΔABC into two parts of equal area. Find the minimum length of DE.

Solution: Area of $\Delta ABC = (5)(12)/2 = 30$, and $\sin A = 5/13$.

Let AD = x, AE = y, then area of $\Delta ADE = (xy \sin A) / 2 = 15$

Therefore $xy = 78$.

By Cosine Law, $DE^2 = x^2 + y^2 - 2xy \cos A$

$$= (x-y)^2 + 2xy(1 - \cos A)$$
$$= (x-y)^2 + 2(78)(1 - 12/13)$$
$$= (x-y)^2 + 12$$

Minimum length of DE = $\sqrt{12}$.

13. (1 mark) D is a point inside ΔABC. PDS, QDT and RDU are lines parallel to BA, CA and CB respectively such that P, Q lie on BC, R, S lie on CA, and T, U lie on AB. If the areas of ΔTUD, ΔPQD and ΔRSD are respectively 8, 128 and 32, find the area of ΔABC.

Solution: Let the Area of $\triangle ABC$ be S.

Note that $\triangle TUD \sim \triangle ABC$, and their areas are in the ratio $UD^2 : BC^2$

Therefore $\sqrt{8} / \sqrt{S} = UD / BC$

Similarly $\sqrt{128} / \sqrt{S} = PQ / BC$

Therefore $\sqrt{32} / \sqrt{S} = DR / BC$

Furthermore, $BC = BP + PQ + QC = UD + PQ + DR$

Therefore $(\sqrt{8} + \sqrt{128} + \sqrt{32}) / \sqrt{S} = (UD + PQ + DR) / BC = 1$

$$\sqrt{S} = 14\sqrt{2}$$
$$S = 392$$

14. (2 marks) The numbers $x_1, x_2, \ldots, x_{2000}$ are such that $|x_1 - x_2| + |x_2 - x_3| + \ldots + |x_{1999} - x_{2000}| = 2000$. Find the largest value of

$|y_1 - y_2| + |y_2 - y_3| + \ldots + |y_{1999} - y_{2000}|$,

where $y_k = \dfrac{x_1 + x_2 + \cdots + x_k}{k}$, for k = 1, 2, …, 2000.

Solution: $|y_k - y_{k+1}| = |(x_1 + x_2 + \ldots + x_k)/k - (x_1 + x_2 + \ldots + x_{k+1})/(k+1)|$

$$= |(x_1 + x_2 + \ldots + x_k - k\, x_{k+1})/k(k+1)|$$
$$\leq (|x_1 - x_2| + 2|x_2 - x_3| + \ldots + k|x_k - x_{k+1}|) / k(k+1)$$

Hence $|y_1 - y_2| + |y_2 - y_3| + \ldots + |y_{1999} - y_{2000}|$

$$\leq |x_1 - x_2|\ (1/1.2 + 1/2.3 + \ldots + 1/1999.2000)$$
$$+ 2|x_2 - x_3|\ (1/2.3 + 1/3.4 + \ldots + 1/1999.2000)$$
$$+ \ldots + 1999|x_k - x_{k+1}|(1/1999.2000)$$
$$= |x_1 - x_2|\ (1 - 1/2000) + |x_2 - x_3|(1 - 2/2000) + \ldots$$
$$+ |x_{1999} - x_{2000}|(1 - 1999/2000)$$
$$\leq 2000(1 - 1/2000)$$
$$= 1999$$

Note that $x_1 = 2000$ and $x_2 = x_3 = \ldots = x_{2000} = 0$ gives the extreme case.