第四屆培正數學邀請賽

4th Pui Ching Invitational Mathematics Competition

決賽（中三組）

Final Event (Secondary 3)

時限：2 小時

Time allowed: 2 hours

參賽者須知：

Instructions to Contestants:

1. 本卷共設甲、乙兩部分，總分爲 100 分。
   This paper is divided into Section A and Section B. The total score is 100.

2. 除特別指明外，本卷內的所有數均爲十進制。
   Unless otherwise stated, all numbers in this paper are in decimal system.

3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。
   Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。
   Put your answers on the spaces provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。
   The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。
   The diagrams in this paper are not necessarily drawn to scale.

甲部（60 分）

Section A (60 marks)

第 1 至第 4 題，每題 3 分。	Questions 1 to 4 each carries 3 marks.
第 5 至第 8 題，每題 5 分。	Questions 5 to 8 each carries 5 marks.
第 9 至第 12 題，每題 7 分。	Questions 9 to 12 each carries 7 marks.

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1. 一張大小爲 24×32 的長方形紙張可以剪成 n 小塊，使得每小塊皆是正方形。（每小塊的大小不一定相同。）求 n 的最小可能值。

   A rectangular piece of paper of size 24×32 can be cut into n small pieces such that each small piece is square in shape. (The small pieces are not necessarily of the same size.) Find the smallest possible value of n.

2. 以下是一段清朝的歷史：

   「乾隆晚年，寵信和珅，和珅則借機貪污。這個和珅竟想打皇上的貢品主意！每有貢品，和珅都先扣起一部分，才將餘下的呈交皇上。而皇上又會將所呈交的貢品的四成賜予和珅。結果，和珅實際所得的貢品，竟比皇上實際所得的多一半！」

   問和珅所扣起的「一部分」，佔原有貢品的幾分之幾？

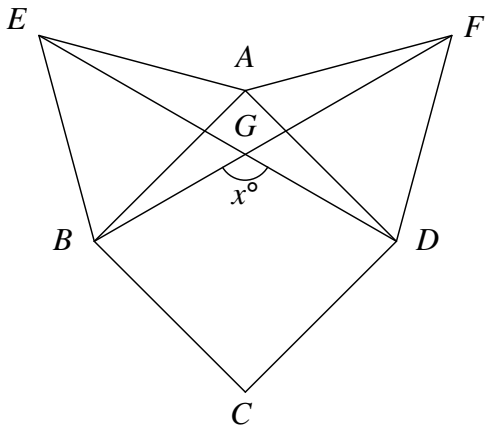
   Here is a piece of history of the Qing dynasty:

   "Emperor Qian-long trusted his official He-shen. He-shen took advantage of this to corrupt. Whenever there were gifts to the emperor, he would keep a part of the gifts himself and pass the remaining gifts to the emperor. The emperor then gave 40% of the remaining gifts to He-shen. The gifts obtained by He-shen turned out to be half more than those actually obtained by the emperor!"

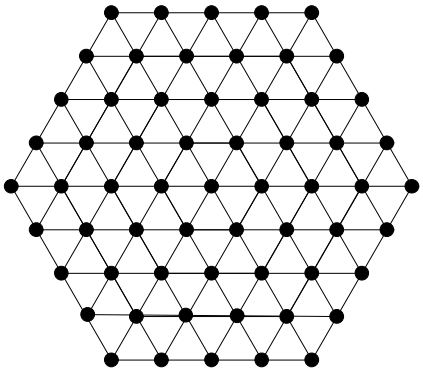
   What fraction of the original gifts was the 'part of the gifts' kept by He-shen before passing the rest to the emperor?

3. 圖中，$ABCD$ 爲正方形，ABE 和 ADF 爲等邊三角形，BF 交 DE 於 G。若 $\angle BGD = x°$，求 x。

   In the figure, $ABCD$ is a square while ABE and ADF are equilateral triangles. BF and DE meet at G. If $\angle BGD = x°$, find x.

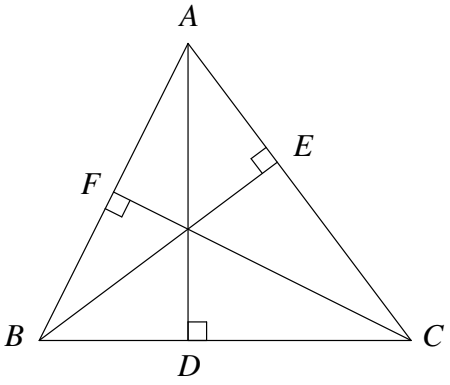


4. 圖中的 61 點排列成 96 個面積爲 1 的等邊三角形。若在這 61 點中選取 3 點，使其成爲一個三角形的頂點，這個三角形的面積最大是多少？



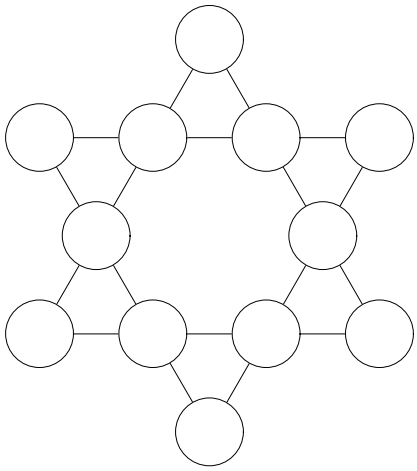
The 61 points in the figure make up 96 equilateral triangles each with area 1. If 3 points are to be chosen from these 61 points so that they form the vertices of a triangle, what is the largest possible area of this triangle?

5. 在 ΔABC 中，$AB = 4$、$BC = 7$、$CA = 5$。AD、BE、CF 是 ΔABC 的三條垂線。求 $\dfrac{AD+BE}{CF}$ 的值。



In ΔABC, $AB = 4$, $BC = 7$ and $CA = 5$. AD, BE and CF are the three altitudes of ΔABC. Find the value of $\dfrac{AD+BE}{CF}$.

6. 小芬在圖中的每個圓圈內分別填上 1 至 12 之間的其中一個整數，並且沒有重覆。她發現，任何四個成一直線的圓圈內的四數之和都不同。這些和之中，設最大的一個爲 M，最小的一個爲 m。求 $M-m$ 的最小值。



Sally put one of the integers from 1 to 12 into each of the circles in the figure without repetition. She found that the sum of any four numbers which lie on the same straight line is different from each other. Let M and m be the largest and smallest of these sums, respectively. Find the smallest value of $M-m$.

7. 在所示的算式中，不同的字母代表不同的數字。已知 S = 5，求四位數 MATH 的值。

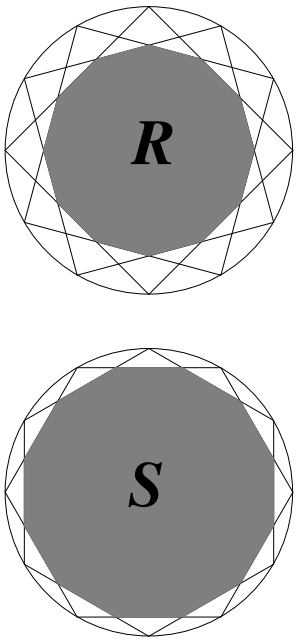
$$\begin{array}{rcccc} & M & A & T & H \\ & & & A & T \\ & P & C & M & S \\ + & & & & 9 \\ \hline A & P & R & I & L \end{array}$$

In the calculation shown, different letters represent different digits. Given that S = 5, find the value of the four-digit integer MATH.

8. 小明和小麗分別就讀於甲、乙兩所學校。某天，甲校其中 42% 的學生轉往乙校，結果乙校的學生人數比甲校多 56%。求乙校原有的學生人數的最小值。

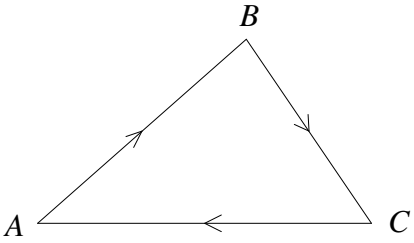
Myron and Lily study in Schools A and B respectively. One day, 42% of the students of School A were transferred to School B. As a result, School B has 56% more students than School A. What is the smallest possible value of the original number of students in School B?

9. 如圖所示，兩人各自在兩個全等的圓形的圓周上標了 12 點將圓周分成 12 等份，再分別用不同的方法連起這些點，最後再把圖案中央的正十二邊形填色。若正十二邊形 R 和 S 的邊長分別是 r cm 和 s cm，求 $\frac{r}{s}$。



As shown in the figure, two people each marks 12 points on the circumference of two identical circles to divide the circumference into 12 equal parts. After that, they connect the points in different ways and then colour the regular 12-sided polygon in the middle. If the side lengths of regular 12-sided polygons R and S are r cm and s cm respectively, find $\frac{r}{s}$.

10. 圖中，AB = 25 m、BC = 20 m、CA = 30 m。小美、比蒂和嘉兒分別從 A 點、B 點和 C 點出發，沿著三角形的邊依順時針方向分別以 x m/s、1.5 m/s 和 2 m/s 的速率散步。若其中兩人相遇，則她們會結伴同行，而步行速度以較慢者爲準。若她們首次三人相遇的地方是 B 點，求 x 的最小值。



In the figure, AB = 25 m, BC = 20 m and CA = 30 m. Amy, Betty and Cathy start at A, B and C respectively, walking along the sides of the triangle in the clockwise direction, with speeds x m/s, 1.5 m/s and 2 m/s respectively. When two of them meet, they will walk together at the lower speed among them. If the first time the three girls meet is at B, find the minimum value of x.

11. 現有整數 1、2、…、50000。若把它們乘起來，最後四位數字是「0000」。最少要拿走多少個整數，才可使餘下的整數之積的最後四位數字爲「2005」？

Now we have the integers 1, 2, …, 50000. If we multiply them together, the last four digits will be '0000'. At least how many integers must be removed so that the product of the remaining integers ends with '2005'?

12. 區先生和五位學生小陳、小李、小張、小王和小何玩遊戲。他準備了 7 張分別寫上了 1、2、3、4、5、6 和 7 的咭片，並在每位學生額前都放一張咭，以及藏起餘下的兩張咭。每人都只可看到別人額前的咭上的數字，但看不見自己額前的咭上的數字。以下是他們於較後時間的對話。

區先生問：「小陳，你知道自己額前的數字嗎？」

「我不知道，但我知道我的額前的數字大於小李和小張的數字。」小陳回答。

區先生問所有學生：「現在有人知道自己額上的數字嗎？」

小李、小張和小王一同答道：「我知道了。」

「可是我們還沒有知道呢。」小陳和小何投訴。

「讓我給你們一點提示吧。」小李說。「你們額前的數字之和是 4 的倍數。」

小陳和小何齊回答：「我們知道了。謝謝小李。」

假設學生都是聰明的（即當他們有足夠的資料便一定可以作出推論），求兩張藏起了的咭上的數字之積。

Mr Au played a game with five students, Ben, Carl, Don, Eric and Fred. He prepared 7 cards with the numbers 1, 2, 3, 4, 5, 6 and 7 written respectively. Mr Au put a card on the forehead of each student. The two remaining cards are hidden. Every student could see the numbers on the cards on others' forehead but not the one on his. Their subsequent conversations are as follows.

'Ben, do you know the number on your forehead?' Mr Au asked.

'No, but I know the number on my forehead is greater than Carl's and Don's,' said Ben.

'Does anyone know the number on your forehead now?' Mr Au asked all students.

'I know.' replied Carl, Don and Eric together.

'But we still do not know what they are.' Ben and Fred complained.

'Let me give you a hint.' said Carl, 'the sum of the numbers on your foreheads is a multiple of 4.'

'We know now. Thanks, Carl.' answered Ben and Fred.

Assume that all students are intelligent (i.e. they can make deductions whenever there is enough information). What is the product of the hidden numbers?

乙部（40 分）

Section B (40 marks)

13. 一位外星人帶了一部特別的鋼琴來訪地球。這台鋼琴有 2005 個琴鍵，分別編號爲 1 號、2 號、…、2005 號。連續的琴鍵之間的距離相同。

An alien visited the Earth with his special piano. This piano has 2005 keys, which are labelled as number 1, number 2, …, number 2005. Consecutive keys are separated by the same distance.

1	2	3	… … … …	2004	2005

(a) 外星人每次按下剛好 k 個琴鍵，按了 101 次後，就把每個琴鍵都按了奇數次（不同琴鍵被按的次數不一定相同）。求 k 最小的可能值。（5 分）

The alien pressed exactly k keys each time for 101 times. As a result of this, each key has been pressed an odd number of times (different keys may be pressed by different numbers of times). What is the smallest possible value of k? (5 marks)

(b) 外星人有 101 隻手指。當他每次以他的 101 隻手指同時按下 101 個不同的琴鍵，且任何兩隻相鄰的手指的距離皆相同時，則我們稱這 101 個琴鍵時所發出的音爲一個「和弦」。這特別的鋼琴可以奏出多少個不同的「和弦」？（5 分）

The alien has 101 fingers. When it uses all its 101 fingers to press 101 different keys simultaneously in a way such that the distances between successive fingers are the same, then we call the sound obtained by pressing these 101 keys a 'chord'. How many 'chords' can this special piano produce? (5 marks)

(c) 爲了表示友好，外星人按下了 m 號及 n 號琴鍵，其中 $(m^2, n^2) = 6(m, n) - 9$，這裡 (x, y) 代表 x 和 y 的最大公因數。求 $m+n$ 的最大可能值。（6 分）

To say hello, the alien pressed two keys which are labelled as number m and number n, where m and n satisfy $(m^2, n^2) = 6(m, n) - 9$. Here (x, y) denotes the H.C.F. of x and y. Find the greatest possible value of $m+n$. (6 marks)

(d) 外星人離開地球前，送了一塊寶石給地球人。這寶石的形狀是一個長方體，大小爲15 厘米×30 厘米×75 厘米 。地球人製作了多個寶石的複製本作展覽之用。每一個複製本均與寶石實物相似，且有一邊長 10 厘米。複製本的體積的所有可能值之和是多少立方厘米？ （4 分）

The alien gave us a gemstone as a present when he left the Earth. The gemstone is in the shape of a cuboid and its dimensions are $15\,\text{cm} \times 30\,\text{cm} \times 75\,\text{cm}$. A number of copies of the gemstone were made for display purposes. Each copy is similar to the original gemstone, and has a side of length 10 cm. Find, in cm^3, the sum of all possible volumes of the gemstone copies. (4 marks)

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14. 本屆培正數學邀請賽中，除個人獎項外，亦會頒發學校團體獎。爲比較參賽者在不同卷別中的表現，我們會先把各參賽者的分數化成「相對得分」，方法如下：

相對得分 = 學生的實際得分 ÷ 同級參賽者中首八名的平均得分

在初賽中，各級的試卷皆有 20 題，其中 3 分題、4 分題、5 分題、6 分題和 7 分題各佔 4 題，滿分爲 100 分。每所學校最多可派 8 名學生參賽，中一至中四級各 2 名。我們以下式計算「學校的初賽得分」：

學校的初賽得分 = 該校學生在初賽的相對得分總和 ÷ 8

In the current Pui Ching Invitational Mathematics Competition, an overall award for schools will be given in addition to individual awards. In order to compare the performance of participants in different papers, the score of each participant will first be converted into a 'relative score' by the following formula:

Relative Score = Actual Score ÷ Average score of the 8 best students in the same paper

In the Heat Events, each paper consists of 20 questions, including four of each of 3-mark, 4-mark, 5-mark, 6-mark and 7-mark questions. The full score is 100. Each school may send at most 8 students to participate, 2 for each of Secondary 1 to Secondary 4. We get a 'School Score for Heats' according to the following formula:

School Score for Heats = Sum of all relative scores of its students in the Heat Events ÷ 8

註：理論上，在計算相對得分的公式中，除數可能等於 0，使相對得分變得沒有意義。然而，只有當某級所有參賽者皆得 0 分時，這個情況才會發生，而這在實際情況中是極不可能的。因此，在制訂比賽章則時，我們已假設了該除數不等於 0。同學回答此題時，亦應作同樣的假設。

Remark. Theoretically, in the formula for computing the relative score, the divisor can be equal to 0 which makes the relative score undefined. However, this will not happen unless all participants in a certain form get 0 mark. This is extremely unlikely in reality. Hence, when enacting the details of the competition, we have assumed that the above denominator will not be equal to 0. Contestants should make the same assumption when answering this question.

(a) 參賽者最高可得的相對得分是多少？ （4 分）

What is the maximum relative score a contestant may achieve? (4 marks)

(b) 在初賽中三組的試卷中，「同級參賽者中首八名的平均得分」有多少個不同的可能值？（注意我們已假設了此數不等於 0。） （4 分）

How many different possible values of the 'average score of the 8 best students in the same paper' are there for the Secondary 3 paper of the Heat Event? (Note that we have assumed that this is not equal to 0.) (4 marks)

(c) 學校的初賽得分的最大可能值是多少？ （6 分）

What is the maximum School Score for Heats that a school may attain? (6 marks)

(d) 學校的初賽得分可以寫成分數。假設某校的初賽得分約至最簡後爲 $\frac{p}{q}$，而且 q 可被 2^k 整除。求 k 的最大可能值。 （6 分）

The School Score for Heats can be written as a fraction. Suppose that the School Score for Heats for a certain school is $\frac{p}{q}$ in lowest terms. If q is divisible by 2^k, find the largest possible value of k. (6 marks)

全卷完

END OF PAPER