香港培正中學第二屆數學邀請賽

Pui Ching Middle School 2nd Invitational Mathematics Competition

團體賽（高級組）

Group Event (Senior Section)

時限：**45** 分鐘

Time allowed: 45 minutes

參賽者須知：

Instructions to Contestants:

1. 本卷共設 20 題，總分為 100 分。

   There are 20 questions in this paper and the total score is 100.

2. 除特別指明外，本卷內的所有數均為十進制。

   Unless otherwise stated, all numbers in this paper are in decimal system.

3. 除特別指明外，所有答案須以數字的真確值表達，並化至最簡。不接受近似值。

   Unless otherwise stated, all answers should be given in exact numerals in their simplest form. No approximation is accepted.

4. 把所有答案填在答題紙指定的空位上。毋須呈交計算步驟。

   Put your answers on the spaces provided on the answer sheet. You are not required to hand in your steps of working.

5. 不得使用計算機。

   The use of calculators is not allowed.

6. 本卷的附圖不一定依比例繪成。

   The diagrams in this paper are not necessarily drawn to scale.

第 1 至第 4 題，每題 3 分。

Questions 1 to 4 each carries 3 marks.

1. 已知一個半球體的體積和總表面積分別是 $n\pi$ 立方厘米和 $n\pi$ 平方厘米。求 n。

   Given that the volume and the total surface area of a hemisphere are $n\pi$ cubic centimetres and $n\pi$ square centimetres respectively. Find n.

2. 求 $\underbrace{55\ldots55}_{2003\text{ 個數字}}$ 除以 12 時的餘數。

   Find the remainder when $\underbrace{55\ldots55}_{2003\text{ digits}}$ is divided by 12.

3. 求具以下性質的 2003 位正整數的數目：除首兩位數字外，其餘所有數字均等於之前兩位數字的差（例如：374312110…）。

   How many 2003-digit positive integers have the property that except for the first two digits, each digit is equal to the difference between its two preceding digits (e.g. 374312110…)?

4. 某次測驗共有兩部份。甲部共有 6 題，乙部共有 8 題，全卷滿分為 100 分。已知甲部平均每題佔分為乙部平均每題佔分的四分之三。甲部共佔多少分？

   In a test there are two sections. There are 6 questions in Section A and 8 questions in Section B, and the full score is 100 marks. It is known that the average score carried by each question in Section A is three-fourths that of each question in Section B. How many marks does Section A carry?

第 5 至第 8 題，每題 4 分。

Questions 5 to 8 each carries 4 marks.

5. 設 $f(x)=ax+b$，當中 a 和 b 為常數。若 $f\left(f(x)\right)=4x+9$，求 $f(0)$ 的最小可能值。

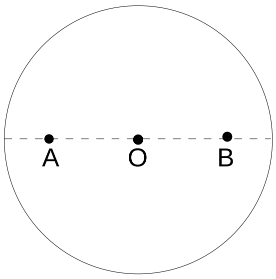
   Let $f(x)=ax+b$ for some constants a and b. If $f\left(f(x)\right)=4x+9$, find the smallest possible value of $f(0)$.

6. 甲、乙二人玩遊戲。甲持有三張分別寫上 1、3、5 的紙牌，乙則持有三張分別寫上 2、4、6 的紙牌。每次各人各自從自己手上的牌中隨意抽出一張，數字較大的一方可得一分。每張紙牌只可出一次，最後得分較多者為勝。問乙勝出的概率為多少？

A game is played between *A* and *B*. *A* has three cards with 1, 3, 5 written on them respectively, while *B* has three cards with 2, 4, 6 written on them respectively. Each time each person selects a card in his hand randomly, and the one whose card has a larger number gets a point. Each card can only be selected once and the one with more points wins. What is the probability that *B* wins?

7. 在一個圓形的湖上有兩個燈塔 *A* 和 *B*。湖的中心 *O* 位於它們的中點，且兩個燈塔的距離是湖的直徑的三分之二。每一每座燈塔的服務範圍定義為湖面上距離此燈塔比距離其他所有燈塔都近的部分。現在政府計劃在湖上設立第三個座燈塔 *C*，使得三個燈塔服務範圍的面積相同。燈塔 *C* 的位置有多少個不同的選擇？

In a circular lake there are two lighthouses *A* and *B*. The centre of the lake *O* lies at their mid-point, and the distance between the lighthouses is two-thirds of the diameter of the lake. The serving area of each lighthouse is defined to be the parts of the lake surface such that it is closest to that lighthouse when compared with any other lighthouses on the lake. Now the government plans to set up a third lighthouse *C* on the lake such that all three lighthouses have the same serving area. How many choices of the position of the third lighthouse *C* does the government have?



8. 設 $[x]$ 代表小於或等於 x 的最大整數，例如：$[2.34] = 2$，$[-2.5] = -3$，$[7] = 7$ 等。有多少個正整數 n 可使得 $\left[\frac{n^2}{3}\right]$ 為質數？

Let $[x]$ denote the greatest integer less than or equal to x. For example, $[2.34] = 2$, $[-2.5] = -3$, $[7] = 7$ and so on. How many positive integers n are there such that $\left[\frac{n^2}{3}\right]$ is prime?

第 9 至第 12 題，每題 5 分。

Questions 9 to 12 each carries 5 marks.

9. 已知 $\frac{1}{13}=0.\dot{0}7692\dot{3}$。 若把 $\frac{23}{130}$ 以小數表示，小數點後首 2003 位數字之和是多少？

   Given $\frac{1}{13}=0.\dot{0}7692\dot{3}$. If $\frac{23}{130}$ is written as a decimal, what is the sum of the first 2003 digits after the decimal point?

10. 一張尺寸為 16×16 的紙被分成 256 個尺寸為 1×1 的小正方形。小怡沿著這些小正方形的邊界，將紙切割成若干面積互不相同的長方形（包括正方形）小塊。她最多可以得到多少小塊？

    A piece of paper of size 16×16 is divided into 256 small squares of size 1×1. Along the sides of the small squares, Emma cuts the paper into rectangular (including square) pieces having pairwise different areas. At most how many pieces can she get?

11. S 是一個直立圓錐體的頂點，AB 為錐體的底直徑，長度為 20。已知 SA = 30，且 C 是 SB 上的一點，使得在圓錐曲面上由 A 到 C 的距離最短。求此最短距離。

    S is the vertex of a right circular cone and AB is its base diameter with length 20. Given that $SA=30$ and C is a point on SB such that the length from A to C on the curved surface of the cone is minimum. Find this minimum length.

12. 設 A = (0, 4)、B = (4, 0) 及 C = (x, y)，當中 x 和 y 為正數，且 $y=-x^2+3x+4$。求 ΔABC 面積的最大可能值。

    Let $A=(0,4)$, $B=(4,0)$ and $C=(x,y)$, where x and y are positive numbers satisfying $y=-x^2+3x+4$. Find the maximum possible area of ΔABC.

第 13 至第 16 題，每題 6 分。

Questions 13 to 16 each carries 6 marks.

13. 某班共有 45 名學生，他們的班號分別為 1、2、⋯、45。現要選 22 名學生參加一項活動，當中被選出的任何兩名學生的班號必須最少相差 2。共有多少種不同的方法選擇學生？

In a class there are 45 students. Their class numbers are 1, 2, ..., 45. Now 22 students are to be chosen to attend an activity, and the class numbers of any two chosen students must differ by at least 2. How many ways are there to choose the students?

14. 若正整數 n 符合以下兩個條件，則稱為「好數」：

(1) n 的最後一位數字不是 0；

(2) 若我們把 n 的各位數字左右倒轉，則所得的數為 n 的倍數。

在 $10 < n < 10000$ 範圍內，共有多少個「好數」？

A positive integer n is said to be 'good' if

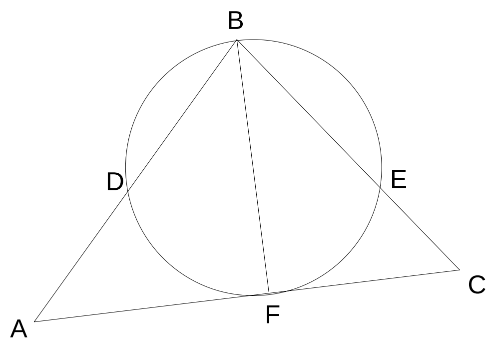
(1) its last digit is not zero; and

(2) when we reverse the digits of n, the number obtained is a multiple of n.

How many 'good' numbers are there in the range $10 < n < 10000$?

15. 如圖所示，BF 為圓形的直徑，$BF \perp AC$，且 $AD = 2BD$。若 $CE = 1$，且 $BE = 3$，求 AD 的長度。

In the figure, BF is the diameter of the circle, $BF \perp AC$ and $AD = 2BD$. If $CE = 1$ and $BE = 3$, find the length of AD.



16. 一個半徑為 r 的球體內有一個四面體。四面體六條邊的邊長分別為 1、3、3、4、4、5。求 r 的最小可能值。

A tetrahedron is contained in a sphere of radius r. The six edges of the tetrahedron have lengths 1, 3, 3, 4, 4, 5. Find the smallest possible value of r.

第 17 至第 20 題，每題 7 分。

Questions 17 to 20 each carries 7 marks.

17. 在 ΔABC 中，$AB = 15$、$BC = 14$、$CA = 13$。X 和 Y 分別為 A 到 BC 及 C 到 AB 的垂足。求 XY 的長度。

    In ΔABC, $AB = 15$, $BC = 14$ and $CA = 13$. X and Y are the feet of the perpendiculars from A to BC and C to AB respectively. Find the length of XY.

18. 某次測驗共有 100 題選擇題，每題答對可得 5 分，答錯扣 2 分，不答得 0 分。合格分數為 100。

    如果一名學生回答 20 題，他必須全部答對方能合格。如果他多答一題，也必須同時答對該題才能合格（因為若他答錯了該題他便只得 98 分）。因此，若以合格為目標，則與其答 21 題，不如只答 20 題。我們說 21 是「壞數」。一般來說，若回答 n 題比回答 $n-1$ 題需要多答對一題才能合格，則我們說 n 是「壞數」。（因此求合格而聰明的學生知道，即使有信心答對 n 題，亦只應回答其中 $n-1$ 題。）

在 $21 \le n \le 100$ 範圍內，有多少個整數 n 是「壞數」？

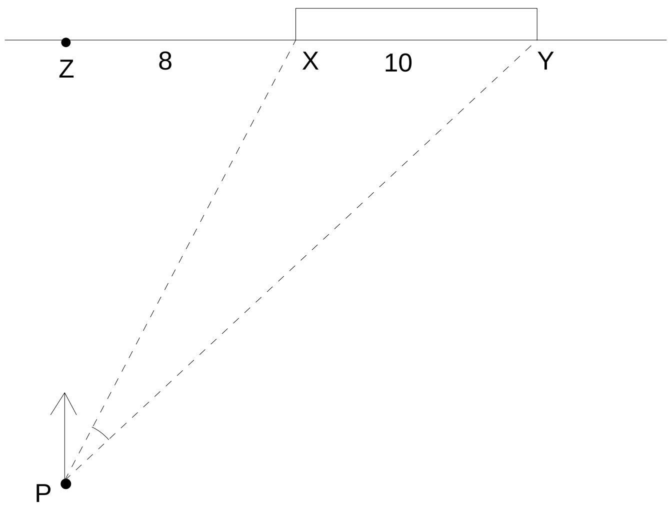
    In a test there are 100 multiple choice questions. For each question, 5 marks will be awarded for a correct answer, 2 marks will be deducted for a wrong answer, and 0 mark will be given if it is left unanswered. The passing mark is 100.

    If a student answers 20 questions, he must get all 20 correct in order to pass. If he answers one more question, he must also get that question correct in order to pass (because if he gets it wrong he will only get 98 marks). So in order to pass, it will be better to answer only 20 rather than 21 questions. We say that 21 is a ‘bad’ number. In general, we say that n is ‘bad’ if answering n questions requires one more correct answer to pass than answering $n-1$ questions. (And those who are intelligent enough and aim to pass will only answer $n-1$ questions even if they are confident with n questions.)

    How many ‘bad’ integers n are there in the range $21 \le n \le 100$?

19. 如圖所示，XY 為足球場上的龍門，闊為 10。Z 為底線上的一點，與 X 的距離為 8。一名球員持球於 P 點，當中 $PZ \perp ZY$，並向正前方推進。當射門角度（即 $\angle XPY$）最大時，球員起腳射門，問此時 $\tan \angle XPY$ 的值是多少？

In the figure, XY is the goal on the football field, with width 10. Z is a point on the touch line at a distance of 8 from X. A player, holding the ball at point P, where $PZ \perp ZY$, proceeds ahead and makes a shot when the angle of shooting (i.e. $\angle XPY$) is the greatest. At this time, what is the value of $\tan \angle XPY$?



20. 一種特別的圓規有四隻腳，其中兩隻是針腳（稱它們為尖 A 及 B），而另外兩隻是鉛筆腳（稱它們為尖 C 及 D）。圓規的設計使得 D 永遠在 ΔABC 的重心。小明用這種特別的圓規在 C 的鉛筆畫一個以 A 為圓心的圓：他先固定 A 及 B，然後讓 C 點在 $AC = 1$ 的限制下移動。若 $AB < 1$，求小明畫圓的同時在 D 的鉛筆所畫出的曲線所圍成的面積。（註：分別把三角形的三個頂點與其對邊的中點以直線連起，則這三條直線交於一點，交點稱為三角形的重心。）

A special kind of compasses has four legs, two of which are pins (call them tips A and B) and the other two are pencils (call them tips C and D). The compass is designed such that D is always at the centroid of ΔABC. Alan uses a pair of compasses of this kind to draw a circle centred at A using the pencil at C by fixing the pins at A and B and then allowing C to move under the constraint $AC = 1$. If $AB < 1$, find the area enclosed by the curve drawn by the pencil at D when Alan draws the circle. (Remark: By joining each vertex of a triangle with the mid-point of its opposite side, the three straight lines meet at a point, called the **centroid** of the triangle.)

全卷完

END OF PAPER

團體賽（高級組）答案

Group Event (Senior Section) Answers

1. $\frac{243}{4}$
2. 7
3. 90
4. 36
5. –9
6. $\frac{5}{6}$
7. 2
8. 2
9. 9016
10. 21
11. $15\sqrt{3}$
12. 8
13. 276
14. 191
15. 4
16. 2.5
17. $\frac{39}{5}$
18. 23
19. $\frac{5}{12}$
20. $\frac{\pi}{9}$