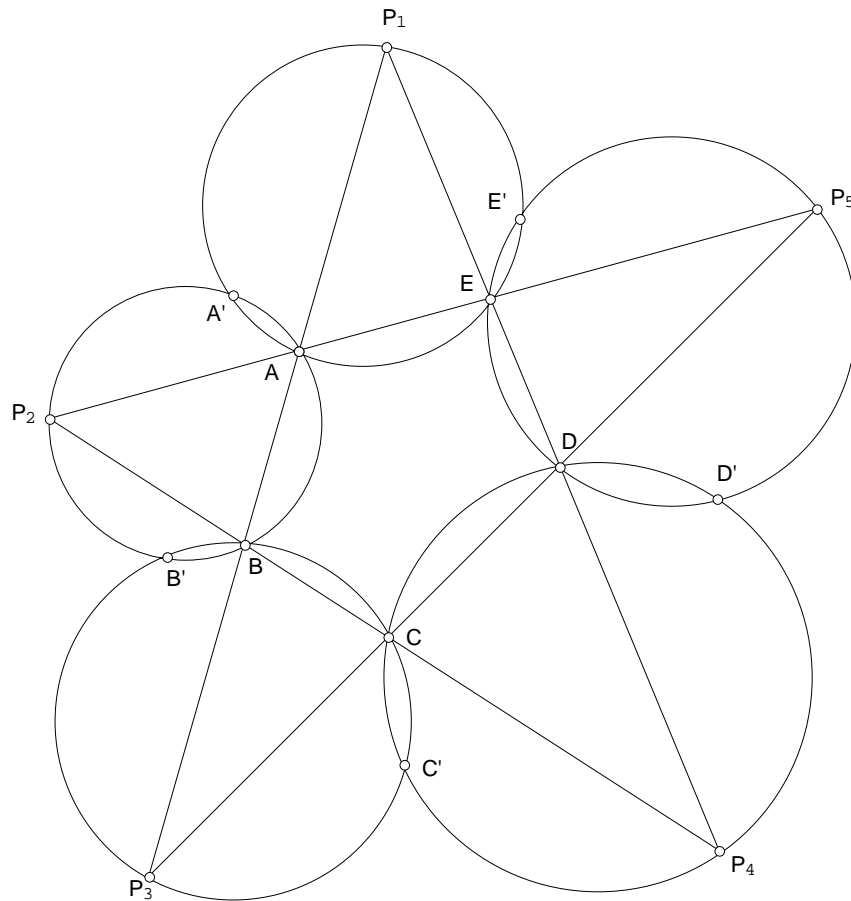


**Problem.**

$ABCDE$  is a convex pentagon. The sides of the pentagon intersect at  $P_1, P_2, P_3, P_4, P_5$  as shown in the following figure. Construct the circumcircles of  $\Delta P_1AE$ ,  $\Delta P_2BA$ ,  $\Delta P_3CB$ ,  $\Delta P_4DC$ , and  $\Delta P_5ED$ . These circumcircles meet at five points  $A', B', C', D', E'$  which different from  $A, B, C, D, E$ . Prove that the points  $A', B', C', D', E'$  are concyclic.



**Solution.**

The trick (and the most difficult part) of the problem is that we observe  $P_1, P_3, C', D, E'$  are concyclic. This can be proved by direct application of *Miquel Theorem*, or simply note that  $\angle P_1P_3C' = \angle P_4CC' = \angle P_4DC'$  (this implies  $P_1, P_3, C', D$  are concyclic and by symmetry  $E'$  lies on the same circle).

Now, we have

$$\begin{aligned} \angle A'B'C' &= \angle A'B'B + \angle BB'C' \\ &= \angle A'AP_1 + \angle BP_3C' \\ &= \angle A'E'P_1 + \angle P_1P_3C' \\ &= \angle P_1E'C' - \angle A'E'C' + \angle P_1P_3C' \end{aligned}$$

Since  $P_1P_3C'E'$  is cyclic quadrilateral,  $\angle P_1E'C' + \angle P_1P_3C' = 180^\circ$ . It follows that

$$\begin{aligned} \angle A'B'C' &= (\angle P_1E'C' + \angle P_1P_3C') - \angle A'E'C' \\ &= 180^\circ - \angle A'E'C' \end{aligned}$$

Hence,  $A', B', C', E'$  are concyclic. Again, by symmetry  $D'$  lies on the circle  $A'B'C'E'$ . This completes the proof.

