Exercise (Inversion)

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1. (Romania, 1997) Let *ABC* be a triangle, *D* a point on side *BC* and ω the circumcircle of *ABC*. Show that the circles tangent to ω, *AD*, *BD* and to ω, *AD*, *DC* are tangent to each other if and only if $\angle BAD = \angle CAD$.

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2. (Russia, 1995) Given a semicircle with diameter *AB* and center *O* and a line which intersects the semicircle at *C* and *D* and line *AB* at *M* ($MB < MA$, $MD < MC$). Let *K* be the second point of intersection of the circumcircles of triangles *AOC* and *DOB*. Prove that angle *MKO* is a right angle.

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3. (USAMO 1993/2) Let *ABCD* be a convex quadrilateral with perpendicular diagonals meeting at *O*. Prove that the reflections of *O* across *AB*, *BC*, *CD*, *DA* are concyclic. (For an added challenge, find a non-inversive proof as well.)

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4. (Apollonius' problem) Given three nonintersecting circles, how many circles are tangent to all three? And how can they be constructed with straightedge and compass?

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5. (IMO 1994 proposal) The incircle of *ABC* touches *BC*, *CA*, *AB* at *D*, *E*, *F* respectively. *X* is a point inside *ABC* such that the incircle of *XBC* touches *BC* at *D* also, and touches *CX* and *XB* at *Y* and *Z* respectively. Prove that *EFZY* is a cyclic quadrilateral.

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6. (Israel, 1995) Let *PQ* be the diameter of semicircle *H*. Circle *O* is internally tangent to *H* and tangent to *PQ* at *C*. Let *A* be a point on *H* and *B* a point on *PQ* such that *AB* is perpendicular to *PQ* and is also tangent to *O*. Prove that *AC* bisects $\angle PAB$.

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7. (IMO 1993/2) Let *A*, *B*, *C*, *D* be four points in the plane, with *C*, *D* on the same side of line *AB*, such that $AC \times BD = AD \times BC$ and $\angle ADB = 90° + \angle ACB$. Find the ratio $(AB \times CD) \div (AC \times BD)$ and prove that the circumcircles of triangles *ACD* and *BCD* are orthogonal.

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8. (Iran, 1995) Let *M*, *N*, *P* be the points of intersection of the incircle of ΔABC with sides *BC*, *CA*, *AB* respectively. Prove that the orthocenter of ΔMNP, the incenter of ΔABC, and the circumcenter of ΔABC are collinear. (The paradigm does not hold here: invert through the incircle, then superimpose the original and inverted diagrams.)

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9. (MOP 1997) Let *ABC* be a triangle and *O* its circumcenter. The lines *AB* and *AC* meet the circumcircle of triangle *BOC* again at B_1 and C_1 respectively. Let *D* be the intersection of lines *BC* and B_1C_1. Show that the circle tangent to *AD* at *A* and having its center on B_1C_1 is orthogonal to the circle with diameter *OD*.