Geometric Inequalities

Notation and Basic Facts

a, b, and c are the sides of ΔABC opposite to A, B, and C respectively.

[ABC] = area of ΔABC

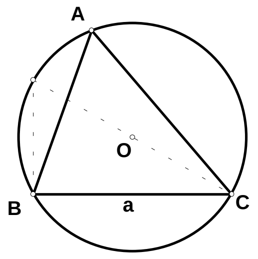
s = semi-perimeter = $\frac{1}{2}(a+b+c)$

r = inradius

R = circumradius

Sine Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Cosine Rule: $a^2 = b^2 + c^2 - 2bc\cos A$



$$[ABC] = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{abc}{4R}$$

$$= \sqrt{s(s-a)(s-b)(s-c)} \qquad \text{(Heron's Formula)}$$

$$= \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} = sr$$

Example 1 Isoperimetric Theorem for Triangle

Among all triangles with a fixed perimeter, the equilateral triangle has the largest area.

Proof:

Using the Heron's Formula and the AM-GM inequality

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} \le \sqrt{s\left[\frac{(s-a)+(s-b)+(s-c)}{3}\right]^3} = \sqrt{s\left(\frac{s}{3}\right)^3} = \frac{s^2}{3\sqrt{3}}$$

with equality holds if and only if s – a = s – b = s– c, i.e a = b = c.

Example 2 [IMO 1961]

Let a, b, c be the sides of a triangle, and T its area. Prove:

$$a^2 + b^2 + c^2 \ge 4\sqrt{3}\,T.$$

In what case does equality hold?

1st solution:

Denote the perimeter of the triangle by p, i.e p = a + b + c, by the isoperimetric theorem for triangle, we have

$$T \le \left(\frac{p}{3}\right)^2 \frac{\sqrt{3}}{4} \qquad \text{with equality holds if and only if } a = b = c \qquad \text{------ (1)}$$

The Cauchy-Schwarz inequality gives

$$p^2 = (a + b + c)^2 \le 3(a^2 + b^2 + c^2) \qquad \text{with equality holds if and only if } a = b = c. \qquad \text{----- (2)}$$

It follows from (1) and (2) that $T \le \frac{a^2 + b^2 + c^2}{3}\frac{\sqrt{3}}{4}$

which is equivalent to $a^2 + b^2 + c^2 \ge 4\sqrt{3}\,T$ with equality holds if and only if a = b = c.

2nd solution:

An equilateral triangle with side c has altitude $\frac{c\sqrt{3}}{2}$. Any triangle with side c will have an altitude perpendicular to c of length $\frac{c\sqrt{3}}{2} + y$. It splits c into parts $\frac{c}{2} - x$ and $\frac{c}{2} + x$. Here x and y are the deviations from an equilateral triangle. Then we have

$$a^2 + b^2 + c^2 - 4\sqrt{3}\,T = \left(\frac{c}{2} - x\right)^2 + \left(\frac{c}{2} + x\right)^2 + 2\left(y + \frac{c}{2}\sqrt{3}\right)^2 + c^2 - 2\sqrt{3}c\left(y + \frac{c}{2}\sqrt{3}\right) = 2x^2 + 2y^2 \ge 0.$$

We have equality if and only if x = y = 0, i.e. when the triangle is equilateral.

Example 3 [IMO 1964]

Suppose a, b, c are the sides of a triangle. Prove that

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \le 3abc.$$

Solution:

Let $x = b + c - a$, $y = c + a - b$, and $z = a + b - c$, then $x, y, z > 0$.

By the AM-GM inequality,

$$\frac{x+y}{2} \ge \sqrt{xy}, \quad \frac{y+z}{2} \ge \sqrt{yz}, \quad \text{and} \quad \frac{z+x}{2} \ge \sqrt{zx}$$

Hence $\frac{x+y}{2}\cdot\frac{y+z}{2}\cdot\frac{z+x}{2} \ge xyz$

which , on substitution, yields $abc \ge (b + c - a)(c + a - b)(a + b - c)$.

The result follows by recognizing that

$$(b + c - a)(c + a - b)(a + b - c) = a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) - 2abc.$$

Example 4 [IMO 1983]

Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \ge 0.$$

Determine when equality occurs.

Solution:

Let $x = b + c - a$, $y = c + a - b$, and $z = a + b - c$, then $x, y, z > 0$.

The given inequality is equivalent to

$$xy^3 + yz^3 + zx^3 \ge xyz(x + y + z) \qquad \text{----- (1)}$$

To prove (1), we use the Cauchy-Schwarz inequality again,

$$(xy^3 + yz^3 + zx^3)(z + x + y) \ge (y\sqrt{xyz} + z\sqrt{xyz} + x\sqrt{xyz})^2 = xyz(x + y + z)^2$$

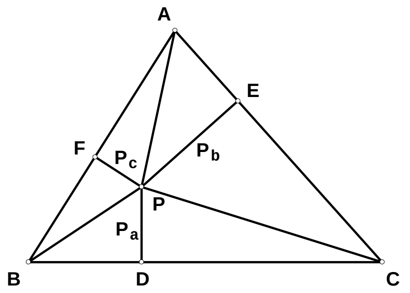
Equality holds if and only if $(xy^3, yz^3, zx^3) = k(z, x, y)$. i.e $x = y = z$.

Example 5 (Erdös-Mordell Inequality)

If P is a point inside triangle ABC and P_a, P_b, P_c are the length of the perpendicular from P to sides a, b, and c respectively, then

$$PA + PB + PC \ge 2(P_a + P_b + P_c).$$

Equality holds if and only if ABC is equilateral and P is the centre.



Solution:

$$DE = \sqrt{P_a{}^2 + P_b{}^2 + 2P_aP_b \cos C} = \sqrt{(P_a \sin B + P_b \sin A)^2 + (P_a \cos B - P_b \cos A)^2} \ge P_a \sin B + P_b \sin A$$

$$PC = \frac{DE}{\sin C} \ge \frac{P_a \sin B + P_b \sin A}{\sin C},$$

Similarly, $PA \ge \frac{P_c \sin B + P_b \sin C}{\sin A}$, $PB \ge \frac{P_c \sin A + P_a \sin C}{\sin B}$

$$PA + PB + PC \ge \frac{P_a \sin B + P_b \sin A}{\sin C} + \frac{P_c \sin B + P_b \sin C}{\sin A} + \frac{P_c \sin A + P_a \sin C}{\sin B}$$

$$= P_a\left(\frac{\sin C}{\sin B} + \frac{\sin B}{\sin C}\right) + P_b\left(\frac{\sin C}{\sin A} + \frac{\sin A}{\sin C}\right) + P_c\left(\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B}\right) \ge 2(P_a + P_b + P_c)$$

(because $x + \frac{1}{x} \ge 2$ for $x > 0$ with equality holds if and only if $x = 1$)

Equality holds if and only if $A = B = C$ and $P_a = P_b = P_c$. i.e. ΔABC is equilateral and P is the centre of it.

Example 6 (Ptolemy's Theorem)

For any point D on the plane of triangle ABC, we have

$$AB\times CD + BC\times AD \geq AC\times BD.$$

Equality holds if and only if A, B, C, D in this order lie on a circle.

Solution:

For any four points z_1, z_2, z_3, z_4 in the plane, we have the identity

$$(z_2 - z_1)(z_4 - z_3) + (z_3 - z_2)(z_4 - z_1) = (z_3 - z_1)(z_4 - z_2).$$

The triangle inequality implies

$$|z_2 - z_1|\,|z_4 - z_3| + |z_3 - z_2|\,|z_4 - z_1| \geq |z_3 - z_1|\,|z_4 - z_2|$$

with equality holds if and only if

$(z_2 - z_1)(z_4 - z_3)$ and $(z_3 - z_2)(z_4 - z_1)$ have the same direction

$\Leftrightarrow$ $(z_2 - z_1)(z_4 - z_3)$ and $(z_2 - z_3)(z_4 - z_1)$ in opposite direction

$\Leftrightarrow$ $\arg\dfrac{(z_2 - z_1)(z_4 - z_3)}{(z_2 - z_3)(z_4 - z_1)} = \pi$

$\Leftrightarrow$ $\arg\dfrac{(z_2 - z_1)}{(z_4 - z_1)} + \arg\dfrac{(z_4 - z_3)}{(z_2 - z_3)} = \pi$

$\Leftrightarrow$ z_1, z_2, z_3, z_4 are either collinear or concyclic.

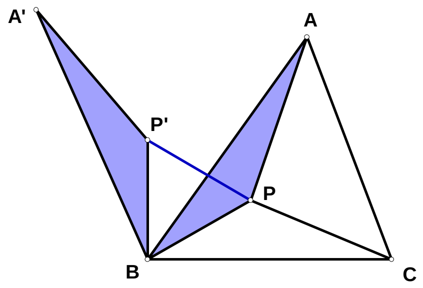
Example 7 (Fermat's Point)

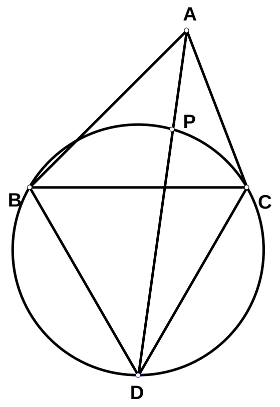
Given ΔABC, find a point P on the plane of ABC such that PA + PB + PC is least.

Solution:

Case (i) Each angle in ΔABC is less than 120^o.

We rotate ΔAPB about B 60^o counterclockwise.





Then $\Delta BPP'$ is equilateral, and so $PA + PB + PC = A'P' + P'P + PC \geq A'C$

To make PA + PB + PC minimum, we choose P such that A′, P′, P and C is collinear.

In this case, $\angle APB = \angle A'P'B = 180^o - \angle BP'P = 120^o$

$\angle BPC = 180^o - \angle P'PB = 120^o$

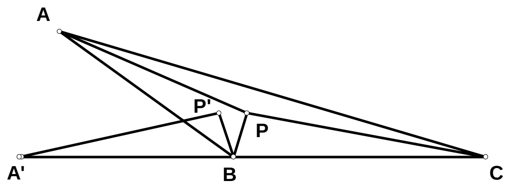
and hence $\angle APC = 360^o - \angle APB - \angle BPC = 120^o$.

To construct such a point P, we erect an equilateral triangle BCD on the side BC.

The point P is the intersecting point of the line AD with the circumcircle of ΔBCD. (Can you see why?)

Case (ii) One of the angles of ΔABC, say B, is greater than or equal to 120^o.

In this case, we rotate ΔAPB about B to $\Delta A'P'B$ counterclockwise such that A′, B and C are collinear.



$\angle ABC \geq 120^o \Rightarrow \angle PBP' \leq 60^o \Rightarrow PP' \leq PB$

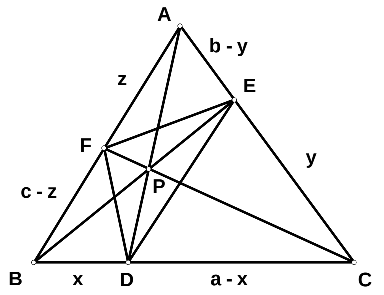
Hence $PA + PB + PC \geq A'P' + PP' + PC \geq A'C = A'B + BC = AB + BC$

It follows that PA + PB + PC is minimum when P = B.

Example 8

For each point P inside a triangle ABC, let D, E and F be the points of intersection of the lines AP, BP, and CP with the sides opposite to A, B, and C respectively. Determine P in such a way that the area of triangle DEF is as large as possible.

Solution:



First, we have [DEF] = [ABC] – [BDF] – [DCE] – [EAF]

With the notations in the figure,

[ABD]/[ABC] = x/a $\Rightarrow$ [ABD] = x[ABC]/a

and [BDF]/[ABD] = (c – z)/c $\Rightarrow$ [BDF] = (c – z)[ABD]/c

Hence, we have $[\mathrm{BDF}] = \frac{(c-z)x}{ac}[\mathrm{ABC}]$.

Similarly, we have $[\mathrm{DCE}] = \frac{(a-x)y}{ba}[\mathrm{ABC}]$ and $[\mathrm{EAF}] = \frac{(b-y)z}{bc}[\mathrm{ABC}]$.

Therefore, $[\mathrm{DEF}] = [\mathrm{ABC}]\left(1-\frac{x(c-z)}{ac}-\frac{y(a-x)}{ba}-\frac{z(b-y)}{cb}\right) = [\mathrm{ABC}](1-u(1-w)-v(1-u)-w(1-v))$,

where $u = x/a$, $v = y/b$, $w = z/c$.

Now, let F = 1 – u(1 – w) – v(1 – u) – w(1 – v) = (1 – u)(1 – v)(1 – w) + uvw.

By Ceva's Theorem, $\frac{a-x}{x}\cdot\frac{c-z}{z}\cdot\frac{b-y}{y}=1 \Rightarrow \left(\frac{1}{u}-1\right)\left(\frac{1}{v}-1\right)\left(\frac{1}{w}-1\right)=1 \Rightarrow (1-u)(1-v)(1-w) = uvw$

It follows that F = 2(1 – u)(1 – v)(1 – w) = 2uvw

$F^2 = 4u(1-u)v(1-v)w(1-w) \le 4\left(\frac{1}{4}\right)^3 = \frac{1}{16} \Rightarrow F \le \frac{1}{4}$

with equality holds if and only if $u = v = w = \frac{1}{2}$.

In conclusion, $\max[\mathrm{DEF}] = \frac{1}{4}[\mathrm{ABC}]$ and is reached if and only if P is the centroid of ΔABC.

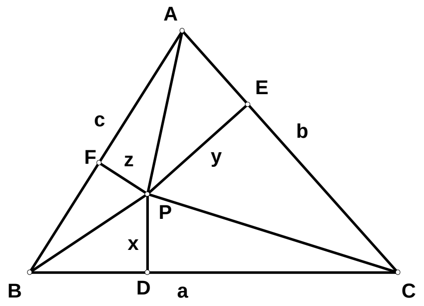
Example 9 **[IMO 1981]**

P is a point inside a given triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P for which

$$\frac{BC}{PD}+\frac{CA}{PE}+\frac{AB}{PF}$$

is least.

Solution:



Using the notations in the figure, 2[ABC] = ax + by + cz ---- (1)

We wish to minimize $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}$ subject to constraint (1).

Using the Cauchy-Schwarz inequality, $(a + b + c)^2 \le (ax + by + cz)\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right) = 2[ABC]\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)$.

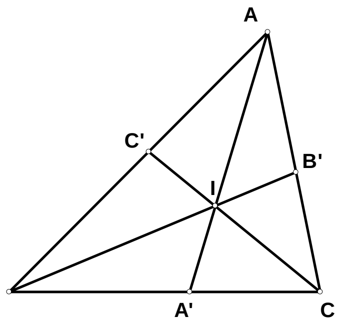
hence $\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right) \ge \frac{(a+b+c)^2}{2[ABC]}$.

The equality holds if and only if (a/x, b/y, c/z) = k(ax, by, cz), i.e. x = y = z.
Thus the minimum value occurs when P is the incentre of the triangle.

Example 10 [IMO 1991]

Given a triangle ABC, let I be the centre of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A′, B′, C′ respectively. Prove that

$$\frac{1}{4} \le \frac{AI\cdot BI\cdot CI}{AA'\cdot BB'\cdot CC'} \le \frac{8}{27} .$$



Solution:

Each inequality holds more generally. Suppose that cevians AA′, BB′, CC′ are concurrent at P which is interior to ABC. Then

$$\frac{AP\cdot BP\cdot CP}{AA'\cdot BB'\cdot CC'} \le \frac{8}{27}$$

Moreover, if P is interior to the medial triangle of ABC (formed by joining the midpoints of the sides) then

$$\frac{AP\cdot BP\cdot CP}{AA'\cdot BB'\cdot CC'} > \frac{1}{4}$$

Proof of the first inequality:

Let [XYZ] denote the area of triangle XYZ. Note that for an arbitrary point P interior to ABC,

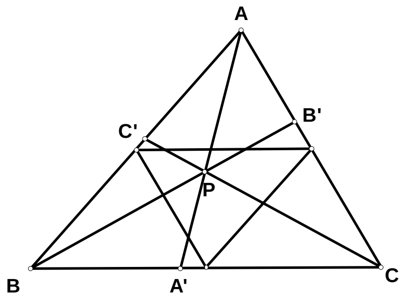
$$\frac{PA'}{AA'} = \frac{[PBC]}{[ABC]},\quad \frac{PB'}{BB'} = \frac{[APC]}{[ABC]},\quad \frac{PC'}{CC'} = \frac{[ABP]}{[ABC]}$$

Thus $\frac{AP}{AA'} = \frac{AA'-PA'}{AA'} = 1-\frac{[PBC]}{[ABC]} = \frac{[ABC]-[PBC]}{[ABC]} = \frac{[ABP]+[APC]}{[ABC]}$, etc.

from which we obtain

$$\frac{AP}{AA'}+\frac{BP}{BB'}+\frac{CP}{CC'} = 2 \qquad \text{----- (*)}$$

The AM-GM inequality $\frac{AP\cdot BP\cdot CP}{AA'\cdot BB'\cdot CC'} \le \left(\frac{2}{3}\right)^3 = \frac{8}{27}$.



Proof of the second inequality:

Clearly, P is interior to the medial triangle of ABC if and only if

$$\frac{AP}{AA'} > \frac{1}{2} ,\quad \frac{BP}{BB'} > \frac{1}{2} ,.\quad \frac{CP}{CC'} > \frac{1}{2} .$$

Suppose that P is interior to the medial triangle and write

$$\frac{AP}{AA'} = \frac{1+x}{2},\quad \frac{BP}{BB'} = \frac{1+y}{2} ,.\quad \frac{CP}{CC'} = \frac{1+z}{2} ,$$

where x, y, z > 0. Note that (*) implies x + y + z = 1.

Thus (1 + x)(1 + y)(1 + z) = 1 + (x + y + z) + (xy + xy + yz) + xyz > 2

and so $\frac{AP\cdot BP\cdot CP}{AA'\cdot BB'\cdot CC'} = \frac{1}{8}\{(1+x)(1+y)(1+y)\} > \frac{1}{4}$

To complete the solution of the original problem, we need to show that $\frac{AP}{AA'} > \frac{1}{2}$, $\frac{BP}{BB'} > \frac{1}{2}$,. $\frac{CP}{CC'} > \frac{1}{2}$ holds for P = I, i.e. the incentre is indeed interior to the medial triangle.

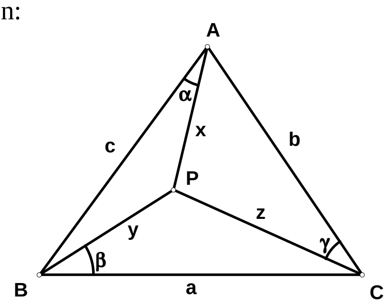
Now $\frac{IA'}{AA'} = \frac{[IBC]}{[ABC]} = \frac{\frac{1}{2}ar}{sr} = \frac{a}{a+b+c} \Rightarrow \frac{AI}{AA'} = 1 - \frac{IA'}{AA'} = \frac{b+c}{a+b+c} > \frac{b+c}{2(b+c)} = \frac{1}{2}$ (since $a < b + c$)

In the same way, $\frac{BI}{BB'} > \frac{1}{2}$,. $\frac{CI}{CC'} > \frac{1}{2}$. The proof is then completed.

Example 11 **[IMO 1991]**

Let ABC be a triangle and P an interior point of ABC. Show that at least one of the angles ∠PAB, ∠PBC, ∠PCA is less than or equal to 30^o.

1st Solution:



Using the result of IMO 1961 question 2 (see Example 2), we have

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\,[ABC] \qquad \text{-----}\ (*)$$

With the notation in the above figure, assume on the contrary that α, β, γ are all greater than 30^o, it follows that each is less than 150^o. Thus for each angle the cosine is less than $\sqrt{3}/2$ and the sine is greater than 1/2.

Now, by cosine rule, $y^2 = c^2 + x^2 - 2cx \cos \alpha$

$z^2 = a^2 + y^2 - 2ay \cos \beta$

$x^2 = b^2 + z^2 - 2bz \cos \gamma$

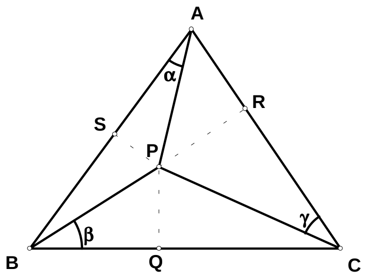
from which we obtain $a^2 + b^2 + c^2 = 2(cx \cos \alpha + ay \cos \beta + bz \cos \gamma) < \sqrt{3}\,(cx + ay + bz)$

On the other hand, $[ABC] = \frac{1}{2}(cx \sin \alpha + ay \sin \beta + bz \sin \gamma) > \frac{1}{4}(cx + ay + bz)$

Combining the two inequality, we get $a^2 + b^2 + c^2 < 4\sqrt{3}\,[ABC]$

which contradicts the inequality (*)

Second solution:



Assume on the contrary that α, β, γ are all greater than 30^o, it follows that each is less than 150^o.

This leads to $PA = PS/\sin \alpha < 2PS$,

$PB = PQ/\sin \beta < 2PQ$,

and $PC = PR/\sin \gamma < 2PR$.

Then $PA + PB + PC < 2(PS + PQ + PR)$

which violates the Erdös-Mordell inequality.

Example 12 [IMO 1995]

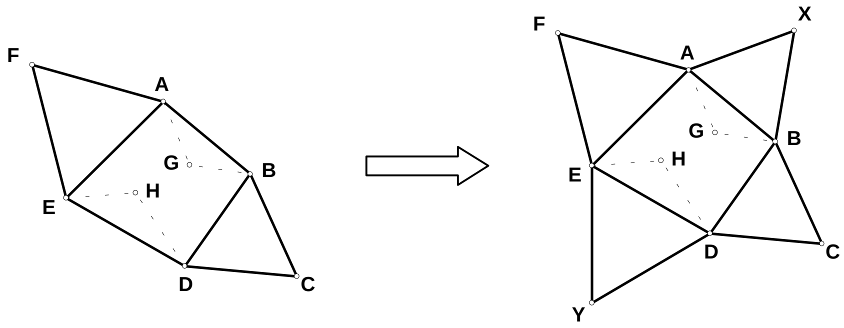
Let ABCDEF be a convex hexagon with

$$AB = BC = CD,$$
$$DE = EF = FA,$$
and $$\angle BCD = \angle EFA = 60^o.$$

Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120^o$. Prove that

$$AG + GB + GH + DH + HE \geq CF.$$

Soln:



As in the figure, we draw equilateral triangles ABX and DEY such that ABCDEF is congruent to DBXAEY.
Since the corresponding sides and angles are equal, CF = XY.
Now $\angle AXB + \angle AGB = 180^o = \angle DYE + \angle DHE$.
So ABXG and DHEY are cyclic quadrilaterals.
By Ptolemy's theorem, $AB \times XG = AX \times GB + XB \times AG$, which implies XG = AG + GB.
Similarly, HY = DH + HE.
Therefore, $AG + GB + GH + DH + HE = XG + GH + HY \geq XY = CF$.