Mathematical Excalibur

Volume 5, Number 5 November 2000 – December 2000

Olympiad Corner

British Mathematical Olympiad, January 2000:

Time allowed: 3 hours 30 minutes

Problem 1. Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects angle PMR.

Problem 2. Show that for every positive integer n,

$$121^n - 25^n + 1900^n - (-4)^n$$

is divisible by 2000.

Problem 3. Triangle ABC has a right angle at A. Among all points P on the perimeter of the triangle, find the position of P such that

$$AP + BP + CP$$

is minimized.

Problem 4. For each positive integer k, define the sequence $\{a_n\}$ by

$$a_0 = 1 \quad \text{and} \quad a_n = kn + (-1)^n a_{n-1}$$

for each $n \geq 1$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is *February 4, 2001*.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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五點求圓錐曲線

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我們知道，圓錐曲線是一些所謂二次形的曲線，即一條圓錐曲線會滿足以下的一般二次方程：$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$，其中 A、B 及 C 不會同時等於 0。假設 $A \neq 0$，那麼我們可以將上式除以 A，並化簡成以下模式：

$$x^2 + bxy + cy^2 + dx + ey + f = 0。$$

以上的方程給了我們一個啟示：就是五點能夠定出一個圓錐曲線。因為如果我們知道了五個不同點的坐標，我們可以將它們分別代入上面的方程中，從而得到一個有 5 個未知數（即 b、c、d、e 和 f）和 5 條方程的方程組。衹要解出各未知數的答案，就可以知道該圓錐曲線的方程了。

不過，上述方法雖然明顯，但真正操作時又困難重重！這是由於有 5 個未知數的聯立方程卻不易解！而且我們在計算之初假設 x^2 的係數非零，但萬一這假設不成立，我們就要改設 B 或 C 非零，並需要重新計算一次了。

幸好，我們可以通過「圓錐曲線族」的想法來解此問題。方法見下例：

例： 求穿過 $A(1, 0)$, $B(3, 1)$, $C(0, 3)$, $D(-4, -1)$, $E(-2, -3)$ 五點的圓錐曲線方程。

解： 利用兩點式，先求出以下各直線的方程：

AB：$\dfrac{y-0}{x-1} = \dfrac{1-0}{3-1}$，即 $x - 2y - 1 = 0$

CD：$\dfrac{y-3}{x-0} = \dfrac{-1-3}{-4-0}$，即 $x - y + 3 = 0$

AC：$\dfrac{y-0}{x-1} = \dfrac{3-0}{0-1}$，即 $3x + y - 3 = 0$

BD：$\dfrac{y-1}{x-3} = \dfrac{-1-1}{-4-3}$，即 $2x - 7y + 1 = 0$

然後將 AB 和 CD 的方程「相乘」，得一條圓錐曲線的方程：

$(x - 2y - 1)(x - y + 3) = 0$，即 $x^2 - 3xy + 2y^2 + 2x - 5y - 3 = 0$。

注意： 雖然上述的方程是一條二次形「曲線」，但實際上它是由兩條直線所組成的。同時，亦請大家留意，該曲線同時穿過 A、B、C 和 D 四點。

類似地，我們又將 AC 和 BD「相乘」，得：

$(3x + y - 3)(2x - 7y + 1) = 0$，即 $6x^2 - 19xy - 7y^2 - 3x + 22y - 3 = 0$。

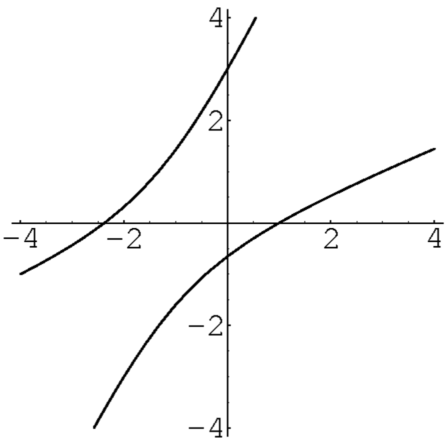
考慮圓錐曲線族：

$x^2 - 3xy + 2y^2 + 2x - 5y - 3 + k(6x^2 - 19xy - 7y^2 - 3x + 22y - 3) = 0$。很明顯，無論 k 取任何數值，這圓錐曲線族都會同樣穿過 A、B、C 和 D 四點。

最後，將 E 點的坐標代入曲線族中，得：$12 + k(-216) = 0$，即 $k = 1/18$，由此得所求的圓錐曲線方程為

$18(x^2 - 3xy + 2y^2 + 2x - 5y - 3) + (6x^2 - 19xy - 7y^2 - 3x + 22y - 3) = 0$，即

$24x^2 - 73xy + 29y^2 + 33x - 68y - 57 = 0$。



Majorization Inequality

Kin Y. Li

The majorization inequality is a generalization of Jensen's inequality. While Jensen's inequality provides one extremum (either maximum or minimum) to a convex (or concave) expression, the majorization inequality can provide both in some cases as the examples below will show. In order to state this inequality, we first introduce the concept of majorization for ordered set of numbers. If

$$x_1 \ge x_2 \ge \cdots \ge x_n,$$
$$y_1 \ge y_2 \ge \cdots \ge y_n,$$
$$x_1 \ge y_1, \quad x_1+x_2 \ge y_1+y_2, \quad \ldots,$$
$$x_1+\cdots+x_{n-1} \ge y_1+\cdots+y_{n-1}$$

and

$$x_1+\cdots+x_n = y_1+\cdots+y_n,$$

then we say $(x_1, x_2, \ldots, x_n)$ *majorizes* $(y_1, y_2, \ldots, y_n)$ and write

$$(x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n).$$

Now we are ready to state the inequality.

Majorization Inequality. *If the function f is convex on the interval I = [a, b] and*

$$(x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n)$$

for $x_i, y_i \in I$, then

$$f(x_1)+f(x_2)+\cdots+f(x_n)$$
$$\ge f(y_1)+f(y_2)+\cdots+f(y_n).$$

For strictly convex functions, equality holds if and only if $x_i = y_i$ *for i = 1, 2, …, n. The statements for concave functions can be obtained by reversing inequality signs.*

Next we will show that the majorization inequality implies Jensen's inequality. This follows from the observation that if $x_1 \ge x_2 \ge \cdots \ge x_n$, then $(x_1, x_2, \ldots, x_n) \succ (x, x, \ldots, x)$, where x is the arithmetic mean of $x_1, x_2, \ldots, x_n$. (Thus, applying the majorization inequality, we get Jensen's inequality.) For $k = 1, 2, \ldots, n-1$, we have to show $x_1+\cdots+x_k \ge kx$. Since

$$(n-k)(x_1+\cdots+x_k)$$
$$\ge (n-k)kx_k \ge k(n-k)x_{k+1}$$
$$\ge k(x_{k+1}+\cdots x_n).$$

Adding $k(x_1+\cdots+x_k)$ to the two extremes, we get

$$n(x_1+\cdots+x_k) \ge k(x_1+\cdots+x_n) = knx.$$

Therefore, $x_1+\cdots+x_k \ge kx$.

Example 1. For acute triangle ABC, show that

$$1 \le \cos A + \cos B + \cos C \le \frac{3}{2}$$

and determine when equality holds.

Solution. Without loss of generality, assume $A \ge B \ge C$. Then $A \ge \pi/3$ and $C \le \pi/3$. Since $\pi/2 \ge A \ge \pi/3$ and

$$\pi \ge A + B (= \pi - C) \ge 2\pi/3,$$

we have $(\pi/2, \pi/2, 0) \succ (A, B, C) \succ (\pi/3, \pi/3, \pi/3)$. Since $f(x) = \cos x$ is strictly concave on $I = [0, \pi/2]$, by the majorization inequality,

$$1 = f\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) + f(0)$$
$$\le f(A) + f(B) + f(C)$$
$$= \cos A + \cos B + \cos C$$
$$\le f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) = \frac{3}{2}.$$

For the first inequality, equality cannot hold (as two of the angles cannot both be right angles). For the second inequality, equality holds if and only if the triangle is equilateral.

Remarks. This example illustrates the equilateral triangles and the degenerate case of two right angles are extreme cases for convex (or concave) sums.

Example 2. Prove that if $a, b \ge 0$, then

$$\sqrt[3]{a+\sqrt[3]{a}} + \sqrt[3]{b+\sqrt[3]{b}} \le \sqrt[3]{a+\sqrt[3]{b}} + \sqrt[3]{b+\sqrt[3]{a}}.$$

(*Source: Math Horizons, Nov. 1995, Problem 36 of Problem Section, proposed by E.M. Kaye*)

Solution. Without loss of generality, we may assume $b \ge a \ge 0$. Among the numbers

$$x_1 = b + \sqrt[3]{b}, \quad x_2 = b + \sqrt[3]{a},$$
$$x_3 = a + \sqrt[3]{b}, \quad x_4 = a + \sqrt[3]{a},$$

x_1 is the maximum and x_4 is the minimum. Since $x_1 + x_4 = x_2 + x_3$, we get $(x_1, x_4) \succ (x_2, x_3)$ or (x_3, x_2) (depends on which of x_2 or x_3 is larger). Since $f(x) = \sqrt[3]{x}$ is concave on the interval $[0, \infty)$, so by the majorization inequality,

$$f(x_4) + f(x_1) \le f(x_3) + f(x_2),$$

which is the desired inequality.

Example 3. Find the maximum of $a^{12} + b^{12} + c^{12}$ if $-1 \le a, b, c \le 1$ and $a + b + c = -1/2$.

Solution. Note the continuous function $f(x) = x^{12}$ is convex on [-1, 1] since $f''(x) = 132x^{10} \ge 0$ on (-1, 1). If $1 \ge a \ge b \ge c \ge -1$ and

$$a + b + c = -\frac{1}{2},$$

then we get $(1, -1/2, -1) \succ (a, b, c)$. This is because $1 \ge a$ and

$$\frac{1}{2} = 1 - \frac{1}{2} \ge -c - \frac{1}{2} = a + b.$$

So by the majorization inequality,

$$a^{12} + b^{12} + c^{12}$$
$$= f(a) + f(a) + f(c)$$
$$\le f(1) + f\left(-\frac{1}{2}\right) + f(-1)$$
$$= 2 + \frac{1}{2^{12}}.$$

The maximum value $2 + (1/2^{12})$ is attained when $a = 1$, $b = -1/2$ and $c = -1$.

Remarks. The example above is a simplification of a problem in the 1997 Chinese Mathematical Olympiad.

Example 4. (1999 IMO) Let n be a fixed integer, with $n \ge 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1 \le i < j \le n} x_i x_j \left(x_i^2 + x_j^2\right) \le C\left(\sum_{1 \le i \le n} x_i\right)^4$$

holds for all real numbers $x_1, x_2, \ldots, x_n \ge 0$.

(b) For this constant C, determine when equality holds.

Solution. Consider the case $n = 2$ first. Let $x_1 = m + h$ and $x_2 = m - h$, then $m = (x_1 + x_2)/2$, $h = (x_1 - x_2)/2$ and

$$x_1 x_2 \left(x_1^2 + x_2^2\right) = 2\left(m^4 - h^4\right)$$
$$\le 2m^4 = \frac{1}{8}(x_1 + x_2)^4$$

with equality if and only if $h = 0$, i.e. $x_1 = x_2$.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *February 4, 2001.*

Problem 116. Show that the interior of a convex quadrilateral with area A and perimeter P contains a circle of radius A/P.

Problem 117. The lengths of the sides of a quadrilateral are positive integers. The length of each side divides the sum of the other three lengths. Prove that two of the sides have the same length.

Problem 118. Let R be the real numbers. Find all functions $f: R \to R$ such that for all real numbers x and y,

$$f(xf(y)+x) = xy + f(x).$$

Problem 119. A circle with center O is internally tangent to two circles inside it at points S and T. Suppose the two circles inside intersect at M and N with N closer to ST. Show that $OM \perp MN$ if and only if S, N, T are collinear. (*Source: 1997 Chinese Senior High Math Competitiion*)

Problem 120. Twenty-eight integers are chosen from the interval [104, 208]. Show that there exist two of them having a common prime divisor.

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Solutions

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Problem 111. Is it possible to place 100 solid balls in space so that no two of them have a common interior point, and each of them touches at least one-third of the others? *(Source: 1997 Czech-Slovak Match)*

Solution 1. **LEE Kai Seng** (HKUST).

Take a smallest ball B with center at O and radius r. Any other ball touching B at x contains a smaller ball of radius r also touching B at x. Since these smaller balls are contained in the ball with center O and radius $3r$, which has a volume 27 times the volume of B, there are at most 26 of these other balls touching B.

Solution 2. **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Consider a smallest ball S with center O and radius r. Let S_i and S_j (with centers O_i and O_j and radii r_i and r_j, respectively) be two other balls touching S at P_i and P_j, respectively. Since r_i, $r_j \ge$ r, we have $O_i O_j \ge r_i + r_j \ge r + r_i = O O_i$ and similarly $O_i O_j \ge O O_j$. So $O_i O_j$ is the longest side of $\Delta O O_i O_j$. Hence $\angle P_iOP_j = \angle O_iOO_j \ge 60^\circ$.

For ball S_i, consider the solid cone with vertex at O obtained by rotating a 30° angle about OP_i as axis. Let A_i be the part of this cone on the surface of S. Since $\angle P_iOP_j \ge 60^\circ$, the interiors of A_i and A_j do not intersect. Since the surface area of each A_i is greater than $\pi(r\sin 30^\circ)^2 = \pi r^2/4$, which is 1/16 of the surface area of S, S can touch at most 15 other balls. So the answer to the question is no.

Other recommended solvers: **CHENG Kei Tsi** (La Salle College, Form 6).

Problem 112. Find all positive integers (x, n) such that $x^n + 2^n + 1$ is a divisor of $x^{n+1} + 2^{n+1} + 1$. *(Source: 1998 Romanian Math Olympiad)*

Solution. **CHENG Kei Tsi** (La Salle College, Form 6), **LEE Kevin** (La Salle College, Form 5) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

For $x = 1$, $2(1^n + 2^n + 1) > 1^{n+1} + 2^{n+1} + 1 > 1^n + 2^n + 1$. For $x = 2$, $2(2^n + 2^n + 1) > 2^{n+1} + 2^{n+1} + 1 > 2^n + 2^n + 1$. For $x = 3$, $3(3^n + 2^n + 1) > 3^{n+1} + 2^{n+1} + 1 > 2(3^n + 2^n + 1)$. So there are no solutions with x = 1, 2, 3.

For $x \ge 4$, if $n \ge 2$, then we get $x(x^n + 2^n + 1) > x^{n+1} + 2^{n+1} + 1$. Now

$$\begin{aligned} & x^{n+1} + 2^{n+1} + 1 \\ &= (x-1)(x^n + 2^n + 1) \\ &\quad + x^n - (2^n+1)x + 3\cdot 2^n + 2 \\ &> (x-1)(x^n + 2^n + 1) \end{aligned}$$

because for $n = 2$, $x^n - (2^n+1)x + 2^{n+1} = x^2 - 5x + 8 > 0$ and for $n \ge 3$, $x^n - (2^n + 1)x \ge x(4^{n-1} - 2^n - 1) > 0$. Hence only $n = 1$ and $x \ge 4$ are possible. In that case, $x^n + 2^n + 1 = x + 3$ is a divisor of $x^{n+1} + 2^{n+1} + 1 = x^2 + 5 = (x-3)(x+3) + 14$ if and only if $x + 3$ is a divisor of 14. Since $x + 3 \ge 7$, $x = 4$ or 11. So the solutions are $(x, y) = (4, 1)$ and (11, 1).

Problem 113. Let $a, b, c > 0$ and $abc \le 1$. Prove that

$$\frac{a}{c} + \frac{b}{a} + \frac{c}{b} \ge a + b + c.$$

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Since $abc \le 1$, we get $1/(bc) \ge a$, $1/(ac) \ge b$ and $1/(ab) \ge c$. By the AM-GM inequality,

$$\frac{2a}{c} + \frac{c}{b} = \frac{a}{c} + \frac{a}{c} + \frac{c}{b} \ge 3\sqrt[3]{\frac{a^2}{bc}} \ge 3a.$$

Similarly, $2b/a + a/c \ge 3b$ and $2c/b + b/a \ge 3c$. Adding these and dividing by 3, we get the desired inequality.

Alternatively, let $x = \sqrt[9]{a^4b/c^2}$, $y = \sqrt[9]{c^4a/b^2}$ and $z = \sqrt[9]{b^4c/a^2}$. We have $a = x^2y$, $b = z^2x$, $c = y^2z$ and $xyz = \sqrt[3]{abc} \le 1$. Using this and the re-arrangement inequality, we get

$$\begin{aligned} &\frac{a}{c} + \frac{b}{a} + \frac{c}{b} = \frac{x^2}{yz} + \frac{z^2}{xy} + \frac{y^2}{zx} \\ &\ge xyz\left(\frac{x^2}{yz} + \frac{z^2}{xy} + \frac{y^2}{zx}\right) = x^3 + y^3 + z^3 \\ &\ge x^2y + y^2z + z^2x = a + b + c. \end{aligned}$$

Problem 114. (*Proposed by Mohammed Aassila, Universite Louis Pasteur, Strasbourg, France*) An infinite chessboard is given, with n black squares and the remainder white. Let the collection of black squares be denoted by G_0. At each moment t = 1, 2, 3, …, a simultaneous change of colour takes place throughout the board according to the following rule: every square gets the colour that dominates in the three square configuration consisting of the square itself, the square above and the square to the right. New collections of black squares G_1, G_2, G_3, … are so formed. Prove that G_n is empty.

Solution. **LEE Kai Seng** (HKUST).

Call a rectangle (made up of squares on the chess board) *desirable* if with respect to its left-lower vertex as origin, every square in the first quadrant outside the rectangle is white. The most crucial fact is that knowing only the colouring of the squares in a desirable rectangle, we can determine their colourings at all later moments. Note that the smallest rectangle enclosing all black squares is a desirable rectangle. We will prove by induction that all squares of a desirable rectangle with at most n black squares will become white by $t = n$. The case $n = 1$ is clear. Suppose the cases $n < N$ are true. Let R be a desirable rectangle with N black squares. Let R_0 be the smallest rectangle in R containing all N black squares, then R_0 is also desirable. Being smallest, the leftmost column and the bottom row of R_0 must contain some black squares. Now the rectangle obtained by deleting the left column of R_0 and the rectangle obtained by deleting the bottom row of R_0 are desirable and contain at most n - 1 black squares. So by $t = n$ - 1, all their squares will become white. Finally the left bottom corner square of R_0 will be white by $t = n$.

Comments: This solution is essentially the same as the proposer's solution.

Other commended solvers: **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Problem 115. (*Proposed by Mohammed Aassila, Universite Louis Pasteur, Stras-bourg, France*) Find the locus of the points P in the plane of an equilateral triangle ABC for which the triangle formed with PA, PB and PC has constant area.

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Without loss of generality, assume $PA \ge PB, PC$. Consider P outside the circumcircle of ΔABC first. If PA is between PB and PC, then rotate ΔPAC about A by 60° so that C goes to B and P goes to P'. Then $\Delta APP'$ is equilateral and the sides of $\Delta PBP'$ have length PA, PB, PC.

Let O be the circumcenter of ΔABC, R be the circumradius and $x = AB = AC = \sqrt{3}AO = \sqrt{3}R$. The area of $\Delta PBP'$ is the sum of the areas of $\Delta PAP'$, ΔPAB, $\Delta P'AB$ (or ΔPAC), which is

$$\frac{\sqrt{3}}{4}PA^2 + \frac{1}{2}PA\cdot x\sin\angle PAB + \frac{1}{2}PA\cdot x\sin PAC .$$

Now

$$\sin\angle PAB + \sin\angle PAC$$
$$= 2\sin 150^\circ \cos(\angle PAB - 150^\circ)$$
$$= -\cos(\angle PAB + 30^\circ)$$
$$= -\cos\angle PAO = \frac{PO^2 - PA^2 - R^2}{2PA\cdot R}.$$

Using these and simplifying, we get the area of $\Delta PBP'$ is $\sqrt{3}(PO^2 - R^2)/4$.

If PC is between PA and PB, then rotate ΔPAC about C by 60° so that A goes to B and P goes to P'. Similarly, the sides of $\Delta PBP'$ have length PA, PB, PC and the area is the same. The case PB is between PA and PC is also similar.

For the case P is inside the circumcircle of ΔABC, the area of the triangle with sidelengths PA, PB, PC can similarly computed to be $\sqrt{3}(R^2 - PO^2)/4$. Therefore, the locus of P is the circle(s) with center O and radius $\sqrt{R^2 \pm 4c/\sqrt{3}}$, where c is the constant area.

Comments: The proposer's solution only differed from the above solution in the details of computing areas.

Olympiad Corner

(continued from page 1)

Problem 4. ***(cont'd)***

Determine all values of k for which 2000 is a term of the sequence.

Problem 5. The seven dwarfs decide to form four teams to compete in the Millennium Quiz. Of course, the sizes of the teams will not all be equal. For instance, one team might consist of Doc alone, one of Dopey alone, one of Sleepy, Happy and Grumpy as a trio, and one of Bashful and Sneezy as a pair. In how many ways can the four teams be made up? (The order of the teams or of the dwarfs within the teams does not matter, but each dwarf must be in exactly one of the teams.)

Suppose Snow White agreed to take part as well. In how many ways could the four teams then be formed?

Majorization Inequality

(continued from page 2)

For the case $n > 2$, let $a_i = x_i/(x_1 + \cdots + x_n)$ for $i = 1, \ldots, n$, then $a_1 + \cdots + a_n = 1$. In terms of a_i's, the inequality to be proved becomes

$$\sum_{1\le i<j\le n} a_i a_j\left(a_i^2 + a_j^2\right) \le C .$$

The left side can be expanded and regrouped to give

$$\sum_{i=1}^{n} a_i^3\left(a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots a_n\right)$$
$$= a_1^3(1-a_1) + \cdots + a_n^3(1-a_n).$$

Now $f(x) = x^3(1-x) = x^3 - x^4 =$ is strictly convex on $\left[0, \frac{1}{2}\right]$ because the second derivative is positive on $\left(0, \frac{1}{2}\right)$.

Since the inequality is symmetric in the a_i's, we may assume $a_1 \ge a_2 \ge \cdots \ge a_n$.

If $a_1 \le \frac{1}{2}$, then since

$$\left(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\right) \succ (a_1, a_2, \ldots, a_n),$$

by the majorization inequality,

$$f(a_1) + f(a_2) + \cdots + f(a_n)$$
$$\le f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + f(0) + \cdots + f(0) = \frac{1}{8}.$$

If $a_1 > \frac{1}{2}$, then $1 - a_1, a_2, \ldots, a_n$ are in $[0, \frac{1}{2}]$. Since

$$(1 - a_1, 0, \ldots, 0) \succ (a_2, \ldots, a_n),$$

by the majorization inequality and case $n = 2$, we have

$$f(a_1) + f(a_2) + \cdots + f(a_n)$$
$$\le f(a_1) + f(1-a_1) + f(0) + \cdots + f(0)$$
$$= f(a_1) + f(1-a_1) \le \frac{1}{8}.$$

Equality holds if and only if two of the variables are equal and the other $n-2$ variables all equal 0.