Mathematical Excalibur

Volume 5, Number 2 March 2000 – April 2000

Olympiad Corner

28th United States of America Mathematical Olympiad, April 1999:

Time allowed: 6 Hours

Problem 1. Some checkers placed on an $n\times n$ checkerboard satisfy the following conditions:

(a) every square that does not contain a checker shares a side with one that does;

(b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2-2)/3$ checkers have been placed on the board.

Problem 2. Let *ABCD* be a cyclic quadrilateral. Prove that

$|AB-CD|+|AD-BC|\geq 2|AC-BD|$.

Problem 3. Let $p>2$ be a prime and let *a, b, c, d* be integers not divisible by *p*, such that

$\{ra/p\}+\{rb/p\}+\{rc/p\}+\{rd/p\}=2$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is May 20, 2000.

For individual subscription for the next five issues for the 00-01 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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漫談質數

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我們從小學開始就已經認識甚麼是質數。一個大於 1 的整數，如果祇能被 1 或自己整除，則我們稱該數為「質數」。另外，我們叫 1 做「單位」，而其他的數字做「合成數」。例如：2、3、5、7…等等，就是質數，4、6、8、9…等等就是合成數。但是除了這個基本的定義之外，一般教科書中，就很少提到質數的其他性質了。而本文就為大家介紹一些與質數有關的人和事。

有人相信，人類在遠古時期，就經已發現質數。不過最先用文字紀錄質數性質的人，就應該是古希臘時代的偉大數學家歐幾里得 (Euclid) 了。

歐幾里得，約生於公元前 330 年，約死於公元前 275 年。他是古代亞歷山大里亞學派的奠基者。他的著作《幾何原本》，集合了平面幾何、比例論、數論、無理量論和立體幾何之大成，一致公認為數學史上的一本鉅著。

《幾何原本》全書共分十三卷，一共包含 465 個命題，當中的第七、八、九卷，主要討論整數的性質，後人又稱這學問為「數論」。第九卷的命題 20 和質數有關，它是這樣寫的：「預先任意給定幾個質數，則有比它們更多的質數。」

歐幾里得原文的證明並不易懂，但改用現代的數學符號，他的證明大致如下：

首先，假如 a、b、c … k 是一些質數。那麼 $abc\cdots k+1$ 或者是質數，或者不是。如果它是質數，那麼就加添了一個新的質數。如果它不是質數，那麼這個數就有一個質因子 p。如果 p 是 a、b、c … k 其中的一個數，由於它整除 $abc\cdots k$，於是它就能整除 1。但這是不可能的，因為 1 不能被其他數整除。因此 p 就是一個新的質數。總結以上兩個情況，我們總獲得一個新的質數。命題得證。

命題 20 提供了一個製造質數的方法，而且可以無窮無盡地製造下去。由此可知，命題 20 實際上是證明了質數有無窮多個。

到了十七世紀初，法國數學家默森 (Mersenne)（1588 – 1648）提出了一條計算質數的「公式」，相當有趣。

因為 $x^n-1=(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)$，所以如果 x^n-1 是質數，$x-1$ 必定要等於 1。由此得 $x=2$。另外，假如 $n=ab$ 並且 $a\leq b$，又令 $x=2^a$，則 $2^n-1=(2^a)^b=x^b-1=(x-1)(x^{b-1}+x^{b-2}+\cdots+x+1)$。所以，如果 2^n-1 是質數，那麼 $x-1$ 必定又要等於 1。由此得 $2^a=2$，即 $a=1$，n 必定是質數。

綜合上述結果，默森提出了一條計算質數的公式，它就是 2^p-1，其中 p 為質數。例如：$2^2-1=3$，$2^3-1=7$，$2^5-1=31$ 等等。但默森的公式祇是計算質數時的「必要」條件，並不是一個「充分」條件；即是說，在某些情況下，由 2^p-1 計算出來的結果，未必一定是質數。例如：$2^{11}-1=2047=23\times 89$，這就不是質數了。因此由默森公式計算出來的數，其實也需要進一步的驗算，才可以知道它是否真正是一個質數。

由於現代的電腦主要用二進數來進行運算，而這又正好和默森公式配合，所以在今天，當人類找尋更大的質數時，往往仍會用上默森的方法。跟據互聯網上的資料，（網址為:

www.utm.edu/research/primes/largest.html），現時發現的最大質數為 $2^{6972593}-1$，它是由三位數學家在 1999 年 6 月 1 日發現的。

默森的好朋友費馬 (Fermat) (1601–1665) 亦提出過一條類似的質數公式。

設 $n = ab$ 並且 b 是一個奇數。令 $x = 2^a$，則 $2^n + 1 = (2^a)^b + 1 = x^b + 1 = (x + 1)(x^{b-1} - x^{b-2} + \cdots - x + 1)$。注意：祇有當 b 為奇數時，上式才成立。很明顯，$2^n + 1$ 並非一個質數。故此，如果 $2^n + 1$ 是質數，那麼 n 必定不能包含奇因子，即 n 必定是 2 的乘冪。換句話說，費馬的質數公式為 $2^{2^n}+1$。

不難驗證，$2^{2^0}+1=3$，$2^{2^1}+1=5$，$2^{2^2}+1=17$，$2^{2^3}+1=257$，$2^{2^4}+1=65537$，它們全都是質數。問題是：跟著以後的數字，又是否質數呢？由於以後的數值增長得非常快，就連費馬本人，也解答不到這個問題了。

最先回答上述問題的人，是十八世紀瑞士大數學家歐拉 (Euler)（1707 – 1783）。 歐拉出生於一個宗教家庭，17 歲已獲得碩士學位，一生都從事數學研究，縱使晚年雙目失明，亦不斷工作，可算是世上最多產的數學家。 歐拉指出，$2^{2^5}+1$ 並非質數。 他的證明如下：

記 $a = 2^7$ 和 $b = 5$。 那麼 $a - b^3 = 3$ 而 $1 + ab - b^4 = 1 + (a - b^3)b = 1 + 3b = 2^4$。 所以

$$\begin{aligned} 2^{32} + 1 &= (2a)^4 + 1 \\ &= 2^4a^4 + 1 = (1 + ab - b^4)a^4 + 1 \\ &= (1 + ab)a^4 + (1 - a^4b^4) \\ &= (1 + ab)(a^4 + (1 - ab)(1 + a^2b^2)), \end{aligned}$$

即 $1 + ab = 641$ 可整除 $2^{32} + 1$，$2^{32} + 1$ 並不是質數！

事實上，到了今天，祇要用一部電子計算機就可以知道：$2^{32} + 1 = 4294967297 = 641 \times 6700417$。 同時，跟據電腦的計算，當 n 大於 4 之後，由費馬公式計算出來的數字，再沒有發現另一個是質數了！ 不過，我們同時亦沒有一個數學方法來證明，費馬質數就祇有上述的五個數字。

自從歐拉證實 $2^{2^5}+1$ 並非質數之後，人們對費馬公式的興趣也隨之大減。 不過到了 1796 年，當年青的數學家高斯發表了他的研究結果後，費馬質數又一再令人關注了。

高斯 (Gauss)（1777 – 1855），德國人。一個數學天才。3 歲已能指出父親帳簿中的錯誤。 22 歲以前，已經成功地證明了多個重要而困難的數學定理。 由於他的天份，後世人都稱他為「數學王子」。

高斯在 19 歲的時候發現，一個正質數多邊形可以用尺規作圖的充分和必要條件是，該多邊形的邊數必定是一個費馬質數！ 換句話說，祇有正三邊形（即正三角形）、正五邊形、正十七邊形、正 257 邊形和正 63357 邊形可以用尺規構作出來，其他的正質數多邊形就不可以了。（除非我們再發現另一個費馬質數。） 高斯同時更提出了一個繪畫正十七邊形方案，打破了自古希臘時代流傳下來，最多祇可構作正五邊形的紀錄。

提到和質數有關的故事，就不可不提「哥德巴赫猜想」了。

哥德巴赫 (Goldbach) 是歐拉的朋友。1742 年，哥德巴赫向歐拉表示他發現每一個不小於 6 的偶數，都可以表示為兩個質數之和，例如： $8 = 3 + 5$、$20 = 7 + 13$、$100 = 17 + 83$ ⋯ 等。哥德巴赫問歐拉這是否一個一般性的現象。

歐拉表示他相信這是一個事實，但他無法作出一個證明。 自此，人們就稱這個現象為「哥德巴赫猜想」。

自從「哥德巴赫猜想」被提出後，經過了整個十九世紀，對這方面研究的進展都很緩慢。 直到 1920 年，挪威數學家布朗 (Brun) 證實一個偶數可以寫成兩個數字之和，其中每一個數字都最多祇有 9 個質因數。 這可以算是一個重大的突破。

1948 年，匈牙利的瑞尼 (Renyi) 證明了一個偶數必定可以寫成一個質數加上一個有上限個因子所組成的合成數。 1962 年，中國的潘承洞證明了一個偶數必定可以寫成一個質數加上一個由 5 個因子所組成的合成數。後來，有人簡稱這結果為 (1 + 5)。

1963 年，中國的王元和潘承洞分別證明了 (1 + 4)。 1965 年，蘇聯的維諾格拉道夫 (Vinogradov) 證實了 (1 + 3)。1966 年，中國的陳景潤就證明了 (1 + 2)。 這亦是世上現時對「哥德巴赫猜想」證明的最佳結果。

陳景潤（1933 – 1996），福建省福州人。 出生於貧窮的家庭，由於戰爭的關係，自幼就在非常惡劣的環境下學習。 1957 年獲得華羅庚的提拔，進入北京科學院當研究員。 在「文化大革命」的十年中，陳景潤受到了批判和不公正的待遇，使他的工作和健康都大受傷害。 1980 年，他當選為中國科學院學部委員。 1984 年證實患上了「帕金遜症」，直至 1996 年 3 月 19 日，終於不治去世。

其實除了對「哥德巴赫猜想」的證明有貢獻外，陳景潤的另一個成就，就是對「孿生質數猜想」證明的貢獻。在質數世界中，我們不難發現有時有兩個質數，它們的距離非常接近，它們的差祇有 2，例如：3 和 5、5 和 7、11 和 13 ⋯ 10016957 和 10016959 ⋯ 等等。所謂「孿生質數猜想」，就是認為這些質數會有無窮多對。而在 1973 年，陳景潤就證得：「存在無窮多個質數 p，使得 $p + 2$ 為不超過兩個質數之積。」

其實在質數的世界之中，還有很多更精彩更有趣的現象，但由於篇幅和個人能力的關係，未能一一盡錄。 以下有一些書籍，內容豐富，值得對本文內容有興趣的人士參考。

參考書目

《數學和數學家的故事》
作者：李學數　出版社：廣角鏡

《天才之旅》
譯者：林傑斌　出版社：牛頓出版公司

《哥德巴赫猜想》
作者：陳景潤　出版社：九章出版社

《素數》
作者：王元　　出版社：九章出版社

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is *May 20, 2000*.

Problem 101. A triple of numbers (a_1, a_2, a_3) = (3, 4, 12) is given. We now perform the following operation: choose two numbers a_i and a_j, $(i \neq j)$, and exchange them by $0.6\,a_i$ - $0.8\,a_j$ and $0.8\,a_i$ + $0.6\,a_j$. Is it possible to obtain after several steps the (unordered) triple (2, 8, 10)? *(Source: 1999 National Math Competition in Croatia)*

Problem 102. Let a be a positive real number and $(x_n)_{n\geq 1}$ be a sequence of real numbers such that $x_1 = a$ and

$$x_{n+1} \geq (n+2)x_n - \sum_{k=1}^{n-1} kx_k \text{ , for all } n \geq 1.$$

Show that there exists a positive integer n such that x_n > 1999! (*Source: 1999 Romanian Third Selection Examination*)

Problem 103. Two circles intersect in points A and B. A line l that contains the point A intersects the circles again in the points C, D, respectively. Let M, N be the midpoints of the arcs BC and BD, which do not contain the point A, and let K be the midpoint of the segment CD. Show that $\angle MKN = 90^\circ$. (*Source: 1999 Romanian Fourth Selection Examination*)

Problem 104. Find all positive integers n such that 2^n-1 is a multiple of 3 and (2^n-1)/3 is a divisor of $4m^2$ + 1 for some integer m. (*Source: 1999 Korean Mathematical Olympiad*)

Problem 105. A rectangular parallelopiped (box) is given, such that its intersection with a plane is a regular hexagon. Prove that the rectangular parallelopiped is a cube. (*Source: 1999 National Math Olympiad in Slovenia*)

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Solutions

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Problem 96. If every point in a plane is colored red or blue, show that there exists a rectangle all of its vertices are of the same color.

Solution. **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7).

Consider the points (x, y) on the co-ordinate plane, where x = 1, 2, …, 7 and y = 1, 2, 3. In row 1, at least 4 of the 7 points are of the same color, say color A. In each of row 2 or 3, if 2 or more of the points directly above the A-colored points in row 1 are also A-colored, then there will be a rectangle with A-colored vertices. Otherwise, at least 3 of the points in each of row 2 and 3 are B-colored and they are directly above four A-colored points in row 1. Then there will be a rectangle with B-colored vertices.

Other recommended solvers: **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **CHEUNG Chi Leung** (Carmel Divine Grace Foundation Secondary School, Form 6), **FAN Wai Tong** (St. Mark's School, Form 7), **LAM Shek Ming Sherman** (La Salle College), **LEE Kar Wai Alvin**, **LI Chi Pang Bill**, **TANG Yat Fai Roger** (La Salle College, Form 5), **LEE Kevin** (La Salle College, Form 4), **LEUNG Wai Ying**, **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 5), **NG Wing Ip** (Carmel Divine Grace Foundation Secondary School, Form 6), **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7), **WONG Wing Hong** (La Salle College, Form 2) and **YEUNG Kai Sing Kelvin** (La Salle College, Form 3).

Problem 97. A group of boys and girls went to a restaurant where only big pizzas cut into 12 pieces were served. Every boy could eat up to 6 or 7 pieces and every girl 2 or 3 pieces. It turned out that 4 pizzas were not enough and that 5 pizzas were too many. How many boys and how many girls were there? (*Source: 1999 National Math Olympaid in Slovenia*).

Solution. **TSE Ho Pak** (SKH Bishop Mok Sau Tseng Secondary School, Form 6).

Let the number of boys and girls be x and y, respectively. Then $7x + 3y \leq 59$ and $6x + 2y \geq 49$. Subtracting these, we get $x + y \leq 10$. Then $6x + 2(10 - x) \geq 49$ implies $x \geq 8$. Also, $7x + 3y \leq 59$ implies $x \leq 8$. So $x = 8$. To satisfy the inequalities then y must be 1.

Other recommended solvers: **AU Cheuk Yin Eddy** (Ming Kei College, Form 7), **CHAN Chin Fei** (STFA Leung Kau Kui College,), **CHAN Hiu Fai** (STFA Leung Kau Kui College, Form 6), **CHAN Man Wai** (St. Stephen's Girls' College, Form 5), **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **CHUNG Ngai Yan** (Carmel Divine Grace Foundation Secondary School, Form 6), **CHUNG Wun Tung Jasper** (Ming Kei College, Form 6), **FAN Wai Tong** (St. Mark's School, Form 7), **HONG Chin Wing** (Pui Ching Middle School, Form 5), **LAM Shek Ming Sherman** (La Salle College), **LEE Kar Wai Alvin, LI Chin Pang Bill, TANG Yat Fai Roger** (La Salle College, Form 5), **LEE Kevin** (La Salle College, Form 4), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **LEUNG Yiu Ka** (STFA Leung Kau Kui College, Form 5), **LYN Kwong To** (Wah Yan College, Form 6), **MOK Ming Fai** (Carmel Divine Grace Foundation Secondary School, Form 6), **NG Chok Ming Lewis** (STFA Leung Kau Kui College, Form 6), **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 5), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **POON Wing Sze Jessica** (STFA Leung Kau Kui College), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4), **WONG Chi Man** (Valtorta College, Form 5), **WONG Chun Ho** (STFA Leung Kau Kui College), **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7), **WONG So Ting** (Carmel Divine Grace Foundation Secondary School, Form 6), **WONG Wing Hong** (La Salle College, Form 2) and **YEUNG Kai Sing Kelvin** (La Salle College, Form 3).

Problem 98. Let ABC be a triangle with $BC > CA > AB$. Select points D on BC and E on the extension of AB such that $BD = BE = AC$. The circumcircle of BED intersects AC at point P and BP meets the circumcircle of ABC at point Q. Show that $AQ + CQ = BP$. (*Source: 1998-99 Iranian Math Olympiad*)

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7) and **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7).

Since $\angle CAQ = \angle CBQ = \angle DEP$ and $\angle AQC = 180^\circ - \angle ABD = \angle EPD$, so $\Delta AQC \sim \Delta EPD$. By Ptolemy's theorem, $BP \times ED = BD \times EP + BE \times DP$. So

$$BP = BD \times \frac{EP}{ED} + BE \times \frac{DP}{ED} =$$

$$AC \times \frac{AQ}{AC} + AC \times \frac{CQ}{AC} = AQ + CQ.$$

Other recommended solvers: **AU Cheuk Yin Eddy** (Ming Kei College, Form 7), **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **LAM Shek Ming Sherman** (La Salle College), **LEE Kevin** (La Salle College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4) and **YEUNG Kai Sing Kelvin** (La Salle College, Form 3).

Problem 99. At Port Aventura there are 16 secret agents. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc., and the last is watching the first. Show that any 11 agents can also be so ordered. (*Source: 1996 Spanish Math Olympiad*)

Solution. **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 5) and **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7).

If some agent watches less than 7 other agents, then he will miss at least 9 agents. The agent himself and these 9 agents will form a group violating the cycle condition. So every agent watches at least 7 other agents. Similarly, every agent is watched by at least 7 agents. (Then each agent can watch at most 15 –7 = 8 agents and is watched by at most 8 agents)

Define two agents to be "connected" if one watches the other. From above, we know that each agent is connected with at least 14 other agents. So each is "disconnected" to at most 1 agent. Since disconnectedness comes in pairs, among 11 agents, at least one, say *X*, will not disconnected to any other agents. Removing *X* among the 11 agents, the other 10 will form a cycle, say

$$X_1, X_2, ..., X_{10}, X_{11} = X_1.$$

Going around the cycle, there must be 2 agents X_i, X_{i+1} in the cycle such that X_i also watches *X* and X_{i+1} is watched by *X*. Then *X* can be inserted to the cycle between these 2 agents.

Other commended solvers: **CHAN Hiu Fai Philip, NG Chok Ming Lewis** (STFA Leung Kau Kui College, Form 6) and **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7).

Problem 100. The arithmetic mean of a number of pairwise distinct prime numbers equals 27. Determine the biggest prime that can occur among them. (*Source: 1999 Czech and Slovak Math Olympiad*)

Solution. **FAN Wai Tong** (St. Mark's School, Form 7) and **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7)

Let $p_1 < p_2 < \cdots < p_n$ be distinct primes such that $p_1 + p_2 + \cdots + p_n = 27n$. Now $p_1 \neq 2$ (for otherwise $p_1 + p_2 + \cdots + p_n$ - 27*n* will be odd no matter *n* is even or odd). Since the primes less than 27 are 2, 3, 5, 7, 11, 13, 17, 19, 23, so $p_n = 27n$ - $(p_1 + \cdots + p_{n-1})$ = 27 + (27 - p_1) $+\cdots+$ (27 - p_{n-1}) ≤ 27+(27-2)+(27-3) $+\cdots+$ (27 –23) = 145. Since p_n is prime, p_n ≤ 139. Since the arithmetic mean of 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 139 is 27. The answer to the problem is 139.

Other recommended solvers: **CHENG Kei Tsi Daniel** (La Salle College, Form 5), **CHEUNG Ka Chung**, **LAM Shek Ming Sherman**, **LEE Kar Wai Alvin**, **TANG Yat Fai Roger**, **WONG Wing Hong**, **YEUNG Kai Sing Kelvin** (La Salle College), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), and **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7).

Olympiad Corner

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Problem 3. ***(cont'd)***
for any integer *r* not divisible by *p*. Prove that at least two of the numbers *a+b, a+c, a+d, b+c, b+d, c+d* are divisible by *p*. (Note: {*x*} = *x* - [*x*] denotes the fractional part of *x*.)

Problem 4. Let $a_1, a_2, ..., a_n (n > 3)$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n$$

and

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, ..., a_n) \geq 2$.

Problem 5. The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

Problem 6. Let *ABCD* be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle *BCD* meets *CD* at *E*. Let *F* be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle *ACF* meet line *CD* at *C* and *G*. Prove that the triangle *AFG* is isosceles.

Interesting Theorems About Primes

Below we will list some interesting theorem concerning prime numbers.

Theorem (due to Fermat in about 1640) *A prime number is the sum of two perfect squares if and only if it is 2 or of the form 4n + 1. A positive integer is the sum of two perfect squares if and only if in the prime factorization of the integer, primes of the form 4n + 3 have even exponents.*

Dirichlet's Theorem on Primes in Progressions (1837) *For every pair of relatively prime integers a and d, there are infinitely many prime numbers in the arithmetic progression a,* $a+d$, $a+2d$, $a+3d$, *…, (In particular, there are infinitely many prime numbers of the form 4n + 1, of the form 6n + 5, etc.)*

Theorem *There is a constant C such that if* p_1, p_2, *…,* p_n *are all the prime numbers less than x, then*

$$\ln(\ln x) - 1 < \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} < \ln(\ln x) + C\ln(\ln(\ln x)).$$

In particular, if $p_1, p_2, p_3, \ldots$ *are all the prime numbers, then*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \cdots = \infty.$$

(The second statement was obtained by Euler in about 1735. The first statement was proved by Chebysev in 1851.)

Chebysev's Theorem (1852) *If* $x > 1$, *then there exists at least one prime number between x and 2x.* (This was known as *Bertrand's postulate* because J. Bertrand verified this for *x* less than six million in 1845.)

Prime Number Theorem (due to J. Hadamard and Ch. de la Vallée Poussin independently in 1896) *Let* $\pi(x)$ *be the number of prime numbers not exceeding x, then*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1.$$

If p_n *is the n-th prime number, then*

$$\lim_{x\to\infty} \frac{p_n}{n\ln n} = 1.$$

(This was conjectured by Gauss in 1793 when he was about 15 years old.)

Brun's Theorem on Twin Primes (1919) *The series of reciprocals of the twin primes either is a finite sum or forms a convergent infinite series, i.e.*

$$\left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\cdots < \infty.$$

As a general reference to these results, we recommend the book *Fundamentals of Number Theory* by William J. Le Veque, published by Dover.