Invariance Principle (May 13, 2000)

1. Given x_0 and y_0 such that $x_0 > y_0 > 0$. Define, for $n = 0,1,2,\cdots$, $x_{n+1} = \dfrac{x_n + y_n}{2}$, $y_{n+1} = \dfrac{2x_n y_n}{x_n + y_n}$. Find the limits of $\{x_n\}$ and $\{y_n\}$.
2. Given x_0 and y_0 such that $y_0 > x_0 > 0$. Define, for $n = 0,1,2,\cdots$, $x_{n+1} = \dfrac{x_n + y_n}{2}$, $y_{n+1} = \sqrt{x_{n+1} y_n}$. Find the limits of $\{x_n\}$ and $\{y_n\}$.
3. The numbers 1, 2, 3, …, 2*n*, where *n* is odd, were originally on the blackboard. Delete two numbers *a*, *b* and replace them by the number $|a-b|$, show that the last remaining number is odd.
4. Starting from the set {3, 4, 12}, take any two numbers *a* and *b* from them and replace these two numbers by $0.6a - 0.8b$ and $0.8a + 0.6b$. After finitely many steps, can we obtain
   (a) {4, 6, 12}
   (b) the set $\{x, y, z\}$ such that $|x-4| < \dfrac{1}{\sqrt{3}}, |y-6| < \dfrac{1}{\sqrt{3}}$ and $|z-12| < \dfrac{1}{\sqrt{3}}$.
5. A circle is divided into 6 equal parts, the numbers 1, 0, 1, 0, 0, 0 are placed into the parts. Choose any two adjacent parts and add 1 to both of them. Continuing this process can we eventually obtain 6 equal numbers?
6. In an island there are 13 blue birds, 15 white birds and 17 red birds. When two birds of different colors encounter, both change into the third color. Can all birds be changed eventually to one color?
7. Each of the numbers $a_1, a_2, \cdots, a_n$ equals to 1 or –1, and it is known that $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = 0$. Show that *n* is divisible by 4.
8. Starting from a set of positive real numbers $S = \{a, b, c, d\}$, replace the set by $\{|a-b|, |b-c|, |c-d|, |d-a|\}$, and so on. Can we eventually obtain {0, 0, 0, 0} if
   (a) the numbers *a*, *b*, *c* and *d* are natural numbers?
   (b) *a*, *b*, *c* and *d* are positive real numbers?
9. In a committee every member has at most three enemies. Show that one can divide the members of the committee into two groups, so that each person has at most one enemy within his group.
10. At the vertices of a regular pentagon are five integers so that their sum exceeds zero. Suppose *x*, *y* and *z* are three adjacent integers such that $y < 0$, then replace *x*, *y*, *z* by $x + y, -y, z + y$. Continue until no negative integers remain. Show that the process will stop after finitely many steps.

11. Let $x_1, x_2, \cdots, x_n$ $(n > 2)$ be n distinct integers. Consider the transformation

$$T(x_1, x_2, \cdots, x_n) = (\frac{x_1 + x_2}{2}, \frac{x_2 + x_3}{2}, \cdots, \frac{x_n + x_1}{2}).$$

Is it possible to find k such that the components of $T^k(X), T^{k+1}(X), \cdots$ are integers? Here $X = (x_1, x_2, \cdots, x_n)$.

12. There are three points A, B and C on the plane. A frog is at point P. In the first step the frog moves to P_1, which is the reflection point of P with respect to A. In the second step the frog moves to P_2, which is the reflection point of P_1 with respect to B. In the third step it moves to P_3, which is the reflection point of P_2 with respect to C. The reflections with respect to A, B, and C are taken again, and so on. Find the distance between P and P_{2000}.

13. Assume an 8×8 chessboard with the usual coloring. You may repaint all squares
    (a) of a row or a column,
    (b) of a 2×2 square.
    Can you eventually obtain a chessboard with one black square?

14. The vertices of an n-gon are labeled by real numbers $x_1, x_2, \cdots, x_n$. Let a, b, c, d be four successive labels. If $(a-d)(b-c) < 0$, then switch b with c. Decide if this process will go on indefinitely.

15. In the table you may switch the signs of all numbers of a row, column, of a parallel to one of the diagonals. Show that at least one "-1" remains.

1	1	1	1
1	1	1	1
1	1	1	1
1	-1	1	1

16. Solve the equation $(x^2 - 3x + 3)^2 - 3(x^2 - 3x + 3) + 3 = x$.

17. Is it possible to transform $f(x) = x^2 + 4x + 3$ into $g(x) = x^2 + 10x + 9$ by a sequence of transformations of the form

$$f(x) \to x^2 f(\frac{1}{x} + 1) \quad \text{or} \quad f(x) \to (x-1)^2 f(\frac{1}{x-1})?$$

18. Does the sequence of squares contain an infinite arithmetic subsequence?

19. Let $d(n)$ be the digital sum of natural number n.
    Solve $n + d(n) + d(d(n)) = 2000$.

20. There is a positive integer in each square of a rectangular table. In each move, you may double each number in a row or subtract 1 from each number of a column. Show that you can reach a table of zeros after finitely many moves.