SEVERAL COMMON COUNTING MODELS

Permutations and combinations are the two most common and most elementary counting models, which readers should be familiar with.

A permutation is to choose r objects ($1 \le r \le n$) from a collection of n objects. We want to count the number of such choices. Here the order by which the objects are chosen is important. That is, we regard two choices to be different if the order of choosing the objects is different, even though the r objects chosen are the same. The number of permutations of r objects from n objects is equal to

$$\boxed{P_r^n = n \times (n-1) \times \cdots \times (n-r+1)}$$

while the number of ways to permute n objects is

$$\boxed{P_n^n = n \times (n-1) \times \cdots \times 2 \times 1 = n!}$$

Here $n!$ is the product of all integers from 1 to n inclusive, read as 'n factorial'.

Combination also concerns the number of ways of choosing r objects from a collection of n objects, but in this case the order by which the r objects are chosen does not matter. Consequently, the formula for computing combinations is given by

$$\boxed{C_r^n = \frac{P_r^n}{r!} = \frac{n \times (n-1) \times \cdots \times (n-r+1)}{r!} = \frac{n!}{r!(n-r)!}}$$

In addition to the permutation and combination models, we frequently come across other counting models. Some counting problems can be solved by applying the permutation and combination models together with the additive and multiplicative rules. But if we adopt other counting models, the calculation can be simplified. The principles behind these counting models are fairly simple, so it is beneficial for us to learn and grasp them. Some common counting models are given below.

Theorem. (Combination with repetition)

The number of ways of choosing *r* objects (with repetition allowed) from a collection of *n* types of objects (each type has at least *r* copies) is given by

$$H_r^n = C_r^{n+r-1} .$$

Proof. We will prove the theorem in two steps.

We first count the number of ways to place *r* identical balls into *n* different boxes, where empty boxes are allowed. Suppose the boxes have x_1, x_2, …, x_n balls respectively, where $x_1 + x_2 + \cdots + x_n = r$. This can be represented as

$$\underbrace{\text{O}\cdots\text{O}}_{x_1 \text{ balls}} + \underbrace{\text{O}\cdots\text{O}}_{x_2 \text{ balls}} + \cdots + \underbrace{\text{O}\cdots\text{O}}_{x_n \text{ balls}}$$

This may be regarded as arranging *r* 'O's and $n-1$ '+'s in a row. Each arrangement corresponds to a way of placing the balls, and there are C_r^{n+r-1} such arrangements (this corresponds to the combination model as we are choosing *r* among $n+r-1$ spaces to place the 'O's).

With this 'placing ball' problem, we see that the equation $x_1 + x_2 + \cdots + x_n = r$ has C_r^{n+r-1} non-negative integer solutions. This is itself a common counting model.

Next we consider combinations with repetitions. Suppose there are *n* different types of objects, a_1, a_2, …, a_n, where each type has at least *r* copies, and we are to choose *r* of them (With repetition allowed). Suppose we have chosen x_1 copies of object a_1, x_2 copies of object a_2, and so on, and x_n copies of object a_n. Consequently we have $x_1 + x_2 + \cdots + x_n = r$, and each non-negative integer solution to the equation corresponds to a choice. So there are altogether C_r^{n+r-1} ways of choosing the objects.

Q.E.D.

In addition to the above counting models, there is yet another common model. This is the model for permutations of objects, some of which may be identical. To make the presentation clear we first consider the simplest situation. Suppose there are *m* white balls and *n* black balls, to be arranged in a row. How many different arrangements are there? This appears to be a problem concerning permutations, but actually it concerns combinations. This is because we are actually choosing *m* places among $m+n$ places to put the white balls (or choosing *n* places to put the black balls). So the number of ways is C_m^{m+n} (or C_n^{m+n}).

Now we try to generalize our conclusion. Suppose we have balls of k different colours, the number of balls of each colour being n_1, n_2, …, n_k respectively, and the total number of balls is m. How many different ways are there to arrange these balls? We can think in this way: we first choose n_1 positions among the m positions to place the balls of the first colour. Then we choose n_2 positions among the remaining $m-n_1$ positions to place the balls of the second colour, and so on. Thus the total number of arrangements is

$$C_{n_1}^{m} \times C_{n_2}^{m-n_1} \times \cdots \times C_{n_k}^{n_k} = \frac{m\ !}{n_1!n_2!\cdots n_k!}.$$

After learning these counting models, we need to learn to apply them to practical problems. First of all we must determine to which counting model the problem belongs, and then apply the related formulae. In dealing with some more complicated problems, we may even need to apply more than one counting models simultaneously.

Example 1.

How many unlike terms are there in the expansion of $(a+b+c)^7$?

Solution.

Each unlike term corresponds to a choice of 7 objects from a collection of 3 types of objects (combination with repetition model). So the number of unlike terms is equal to $H_7^3 = C_7^9 = C_2^9 = 36$.

Example 2.

How many positive integer solutions are there to the equation $x_1 + x_2 + \cdots + x_{10} = 100$?

Solution.

Imagine that 100 identical balls are listed in a row. There are 99 spaces between adjacent balls. We choose among these 9 spaces to put the symbol '+', and each choice corresponds to a solution to the equation. For instance, each solution of the equation can be represented as

$$\underbrace{\text{O}\cdots\text{O}}_{x_1 \text{ balls}} + \underbrace{\text{O}\cdots\text{O}}_{x_2 \text{ balls}} + \cdots + \underbrace{\text{O}\cdots\text{O}}_{x_{10} \text{ balls}}$$

So the number of positive integer solutions to the equation is equal to C_9^{99}.

Alternative Solution.

Let $x_i = y_i + 1$ (i = 1, 2, …, 10). Consequently each positive integer solution to the equation $x_1 + x_2 + \cdots + x_{10} = 100$ corresponds to a non-negative integer solution to $y_1 + y_2 + \cdots + y_{10} = 90$ (the number of which has already been counted). Thus the answer is $H_{90}^{10} = C_{90}^{99} = C_9^{99}$.

Example 3.

Four different dice are thrown. In how many different situations will a sum not exceeding 9 occur?

Solution.

Let x, y, z, u be the four numbers on the dice. Then each situation in which the sum of the numbers on the four dice does not exceed 9 corresponds to a positive integer solution to the inequality $x + y + z + u \le 9$, and to a non-negative integer solution to the inequality $x' + y' + z' + u' \le 5$. Introducing a non-negative variable t, the inequality can be transformed to the equation $x' + y' + z' + u' + t = 5$, which has $H_5^5 = C_5^9 = C_4^9 = 126$ non-negative integer solutions. This is also the answer to the original problem.

Example 4.

There are n different books to be given to r people. They are to get k_1, k_2, …, k_r books respectively, where $k_1 + k_2 + \cdots + k_r = n$. In how many different ways can this be done?

Solution.

This precisely is the model for permutations with some identical objects. So the number of ways is equal to $\dfrac{n\,!}{k_1!k_2!\cdots k_r!}$.

Example 5.

Teams A and B each send 7 members to compete in a chess competition. Member 1 of both teams first compete. The loser is eliminated and the winner proceeds to compete with member 2 of the other team. This process is continued until all members of one team are eliminated. In how many different ways can the competition proceed?

Solution.

Consider the case where 7 A's and 7 B's are arranged in a row. Each letter represents a member of the respective teams in the preassigned order. Each arrangement of letters thus corresponds to a way in which the competition proceeds. For example, the permutation *ABBABBBAAAABAB* corresponds to the situation in which member 1 of Team *A* is defeated by member 1 of Team *B*, and then member 2 Team *A* win two consecutive matches, and so on.

From this, we see that the number of ways is equal to C_7^{14}.