Logistic Equation

Exponential Growth.

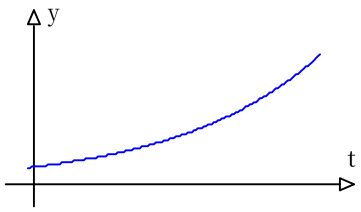
Let $y = \phi(t)$ be the population of a given species at time *t*. The simplest hypotheses concerning the variation of population is that the rate of change of *y* is proportional to the current value of *y*, that is,

$$\frac{dy}{dt} = ry, \quad \text{------------------------------------} \quad (1)$$

where *r* is called the **rate of growth or decline**, depending on whether it is positive or negative.

Here we assume that $r > 0$, so that the population is growing. Solving (1) subject to the initial condition $y(0) = y_0$, we obtain

$$y = y_0 e^{rt}$$



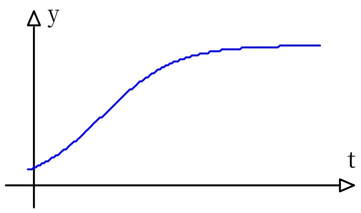
However, it is clear that such ideal conditions cannot continue indefinitely; eventually, limitations on space, food supply, or other resources will reduce the growth rate and bring an end to uninhibited exponential growth.

Logistic Growth.

To take account of the fact that the grown rate actually depends on the population, we replace the constant *r* by $r - ay$, where *a* is also a positive constant. The equation

$$\frac{dy}{dt} = (r - ay)y \quad \text{---------------------------------} \quad (2)$$

is known as the Verhulst equation or the **logistic equation**. The following graph shows the solution of (2) tends to a horizontal line which is called **asymptotically stable solution**.



We see that equation (2) is separable, but our main object is to show how we can obtain important information directly from the differential equation, without solving the equation.

It is often convenient to write the logistic equation in the equivalent form

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y \qquad \text{---------------------------------} \qquad (3)$$

where $K = \frac{r}{a}$, the constant r is called the **intrinsic growth rate**, that is, the growth rate in the absence of any limiting factors.

We first seek solution of (3) of the simplest possible type, that is, constant functions. For such a function,

$$r\left(1 - \frac{y}{K}\right)y = 0\,,$$

hence $y = 0$ or $y = K$ are solution of (3). These solutions are called **equilibrium solutions**, the numbers 0 and K are called **critical points**.

- If $y(0) > 0$, then the solution approach K as $t \to \infty$. If $y(0) < 0$, then it decreases without bound.

- If $y(0) \neq K$, then the solution does not reach the line $y = K$ at any finite time. Similarly, if $y(0) \neq 0$, then $y(t) \neq 0$ for all t. (The fundamental existence and uniqueness theorem guarantees that two different solutions never pass through the same point.)

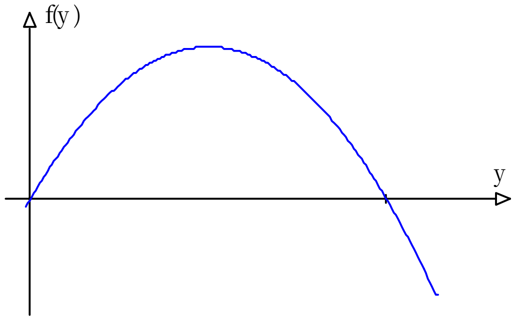
- Except the equilibrium solutions, $\frac{dy}{dt}$ always non-zero, i.e. any non-equilibrium solution always (strictly) increasing or always (strictly) decreasing, depends on the initial value $y(0)$.

Proof.

Let $f(y) = r\left(1 - \frac{y}{K}\right)y$ be a quadratic function in y (note that $f(y) = \frac{dy}{dt}$).

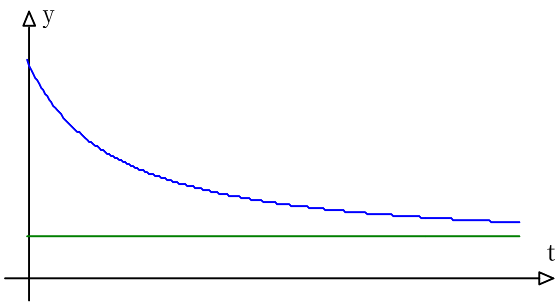
For the case $y(0) > K$, we have $y(t) > K$ for all t (by fundamental existence and uniqueness theorem). So $f(y)$ is always negative and $y(t)$ is strictly decreasing and bounded by K. We conclude that $y_\infty = \lim_{t \to \infty} y(t)$ exists, it remains to show $y_\infty = K$.

Note that $y''(t) = \frac{df}{dy} \times \frac{dy}{dt} = f'(y) \times f(y)$ which is positive since $y(t) > K$, follows that $y'(t)$ is strictly increasing.



Now, we have shown that $y'(t)$ is increasing and bounded above (by zero), which implies $\lim_{t \to \infty} y'(t)$ exists. In fact $\lim_{t \to \infty} y'(t) = 0$, because if it is nonzero then it must be negative and follows that $y(t)$ decreases with bound, which is impossible.

Now, $\lim_{t \to \infty} y'(t) = 0 \Rightarrow \lim_{t \to \infty} f(y) = 0 \Rightarrow f(y_\infty) = 0 \Rightarrow y_\infty = 0$ or K. Since $y(t) > K$ for all *t*, it is impossible that $y_\infty = 0$. This proved $y_\infty = K$.



Similar argument can be applied to the case $0 < y(0) < K$, which we omitted here.