

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Now, we are going to consider the differential equations with the form

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x),$$

where a, b, c are constants and $f(x)$ is a function of x .

Special Case : $f(x) = 0$

In this case, the equation become $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$.

We claim that $y = e^{\lambda x}$ is a solution, put it into the differential equation and we will find that λ satisfy the equation

$$a\lambda^2 + b\lambda + c = 0,$$

this is called *auxiliary equation* or *characteristic equation*.

Since the equation $a\lambda^2 + b\lambda + c = 0$ may have two real roots, repeated root or two complex roots (of course, the term “complex” here not including real case), our consideration should also divide into three cases.

(i) The roots are real and distinct, says λ_1, λ_2

The general solution of the original differential equation is

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x},$$

where A, B are arbitrary constants.

(ii) The roots are equal (i.e. $b^2 - 4ac = 0$), says λ

The general solution of the original differential equation is

$$y = Ae^{\lambda x} + Bxe^{\lambda x},$$

where A, B are arbitrary constants.

(iii) The roots are imaginary, says $u + vi, u - vi$ ($v \neq 0$)

The general solution of the original differential equation is

$$y = e^{ux} (A \cos vx + B \sin vx),$$

where A, B are arbitrary constants.

Sometimes, we may write the general solution into the form

$$y = Ce^{ux} \cos(x + \alpha) \quad \text{or} \quad y = Ce^{ux} \sin(x + \beta),$$

where C, α, β are arbitrary constants.

General Case : $f(x) \neq 0$

For the differential equation $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$, the general solution is sum of two parts, one is called the **complementary function** (餘函數) y_c and the other is the **particular integral** (特別積分) y_p . That is, the general solution is in the form

$$y = y_c + y_p,$$

where y_c is the general solution of $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$ and y_p depends on what $f(x)$ is. For y_c , we have already known how to find it. So, we consider y_p .

- (i) If $f(x)$ is a polynomial of degree r , then we try y_p be a polynomial of degree r . Note that this method fails if $c = 0$, so in that case y_p should be a polynomial with degree $r + 1$ by multiplying a factor x to it.
- (ii) If $f(x)$ is in the form of he^{kx} (where h, k are given constants), then we try $y_p = pe^{kx}$, where p is a constant to be determined. This method fails if e^{kx} appears in the complementary function y_c , in that case we try $y_p = px e^{kx}$. If this fails for a similar reason, we try $y_p = px^2 e^{kx}$.
- (iii) If $f(x)$ is in the form of $m \cos kx + n \sin kx$ (where m, n are given constants), we try $y_p = p \cos kx + q \sin kx$, where p, q are constants to be determined. This method fails if either $\cos kx$ or $\sin kx$ appears in the complementary function y_c , in that case we try $y_p = px \cos kx + qx \sin kx$.
- (iv) If $f(x)$ is sum of different types mentioned above, then we try y_p be the sum of the corresponding types.