FIRST ORDER DIFFERENTIAL EQUATIONS

Type 1. First order differential equations with separable variables

Now, our consideration is given to the case in which

$$\frac{dy}{dx} = f(x) \cdot g(y),$$

where $f(x)$, $g(y)$ are respectively functions of x and y. This equation can be written

$$\frac{dy}{g(y)} = f(x)dx$$

and integration with respect to x gives

$$\int \frac{dy}{g(y)} = \int f(x)dx.$$

The general solution can be found by this way.

Type 2. The linear first order differential equations

The general linear differential equation of the first order can be written

$$\frac{dy}{dx} + f(x)y = g(x),$$

where $f(x)$ and $g(x)$ are functions of x (but not y). A solution can be obtained by multiplying the equation by an ***integrating factor*** μ which makes the left-hand side of the equation an exact differential. The integrating factor μ is given by

$$\ln \mu = \int f(x)dx.$$

After multiplying the integrating factor, the differential equation can be written

$$\frac{d}{dx}(y\mu) = g(x)\mu,$$

the general solution can be found.

Type 3. First order differential equation reducible to linear form

The linear equation in type 2 is a particular case $(n = 0)$ of the more general form

$$\frac{dy}{dx} + f(x)y = g(x)y^n .$$

This equation (oftem known as ***Bernoulli's equation***) can be reduced to a linear differential equation in z and x by means of the substitution

$$y^{1-n} = (1-n)z .$$

With this substitution we have $\frac{dy}{dx} = y^n \frac{dz}{dx}$, and the equation becomes, after division by y^n and replacing y^{1-n} by $(1-n)z$,

$$\frac{dz}{dx} + (1-n) f(x)z = g(x) .$$

This is a linear first order differential equation which we have discussed before.

Type 4. Homogeneous first order differential equations

A first order differential equation is said to be ***homogeneous*** if it can be written in the form

$$\frac{dy}{dx} = f(\frac{y}{x}) .$$

By writing $y = ux$ (note that $\frac{dy}{dx} = u + x\frac{du}{dx}$), the equation becomes

$$x\frac{du}{dx} = f(u) - u .$$

If $f(u) \neq u$, it has reduced to type 1.

If $f(u) = u$ for $u = u_i$ ($i = 1, 2, \cdots, k$), then $y = u_i x$ are also solutions.

Type 5. First order differential equation reducible to standard form

Finally, we are going to consider the equations of the type

$$\frac{dy}{dx} = f(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}) ,$$

where a_1, b_1, c_1, a_2, b_2, c_2 are constants, can be reduced to homogeneous form by regarding x and y as rectangular coordinates and <u>translating the origin of coordinates to the point of intersection</u> of the two straight lines $a_1 x + b_1 y + c_1 = 0$, $a_2 x + b_2 y + c_2 = 0$. This method fails, of course, if the lines are parallel but in this case the equation can be reduced to the first order differential equation with separable variables by the substitution $z = a_1 x + b_1 y$ or $z = a_2 x + b_2 y$.