L' Hospital's Rule

In the following we are going to establish the L' Hospital Rule. For convenience, we will consider right-hand limits at a point *a*; left-hand limits and two-sided limits are treated in exactly the same way. Let $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$ be the extended real number system.

A Preliminary Result.

Let *f* and *g* be defined on $[a,b]$, let $f(a) = g(a) = 0$, and let $g(x) \neq 0$ for $a < x < b$. If *f* and *g* are differentiable at *a* and if $g'(a) \neq 0$, then

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \text{ exists in } \mathbb{R}.$$

Proof.

The result follows immediately from

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}.$$

Before we state and prove the L' Hospital Rule, we need a more general version of the Mean Value Theorem due to Cauchy.

Cauchy Mean Value Theorem.

Let *f* and *g* be continuous on $[a,b]$ and differentiable on (a,b), and assume that $g'(x) \neq 0$ for all *x* in (a,b). Then there exists *c* in (a,b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof.

Follows from Rolle's Theorem we have $g(a) \neq (b)$ since $g'(x) \neq 0$ for all *x* in (a,b). Now, define

$$h(x) = \frac{f(b) - f(a)}{g(b) - g(a)}\big(g(x) - g(a)\big) - \big(f(x) - f(a)\big).$$

Then *h* is continuous on $[a,b]$, differentiable on (a,b), and $h(a) = h(b) = 0$. Therefore, it follows from Rolle's Theorem that there exists a point *c* in (a,b) such that $h'(c) = 0$, i.e.

$$\frac{f(b) - f(a)}{g(b) - g(a)} g'(c) - f'(c) = 0.$$

Dividing by $g'(c) \neq 0$, we obtain the desired result.

L' Hospital Rule I.

Let $-\infty \le a < b \le \infty$ and let *f*, *g* be differentiable on (a,b) such that $g'(x) \ne 0$ for all *x* in (a,b). Suppose that $\lim\limits_{x\to a+} f(x) = \lim\limits_{x\to a+} g(x) = 0$.

If $\lim\limits_{x\to a+} \dfrac{f'(x)}{g'(x)} = L$ exists in $\mathbb{R}^*$, then $\lim\limits_{x\to a+} \dfrac{f(x)}{g(x)} = L$.

Proof.

If α and β are chosen so that $a < \alpha < \beta < b$, then $g(\alpha) \ne g(\beta)$. By the Cauchy Mean Value Theorem, there exists $u \in (\alpha, \beta)$ such that

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(u)}{g'(u)}.$$

We first consider the case $L \in \mathbb{R}$. If $\varepsilon > 0$ is given, there exists $c \in (a,b)$ such that

$$L - \varepsilon < \frac{f'(u)}{g'(u)} < L + \varepsilon \qquad \text{for } u \in (a,c).$$

So

$$L - \varepsilon < \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} < L + \varepsilon \qquad \text{for } a < \alpha < \beta \le c.$$

Take limit as $\alpha \to a+$, we have $L - \varepsilon \le \dfrac{f(\beta)}{g(\beta)} \le L + \varepsilon$ for $a < \beta \le c$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim\limits_{x\to a+} \dfrac{f(x)}{g(x)} = L$.

Similarly, if $L = \infty$ and if $M > 0$ is given, there exists $c \in (a,b)$ such that

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} > M \qquad \text{for } a < \alpha < \beta \le c.$$

Again, we take limit as $\alpha \to a+$, follows that $\dfrac{f(\beta)}{g(\beta)} \ge M$ for $a < \beta \le c$. Since $M > 0$ is arbitrary, we conclude that $\lim\limits_{x\to a+} \dfrac{f(x)}{g(x)} = \infty$.

For the case $L = -\infty$, the argument is the same.

- The converse of L' Hospital Rule I is not true. Even if *f* and *g* satisfy the assumptions and $\lim\limits_{x\to a+} \dfrac{f(x)}{g(x)} = L$, it is possible that $\lim\limits_{x\to a+} \dfrac{f'(x)}{g'(x)}$ does not exist. Here is an example:

Let $f(x) = x^2 \sin\frac{1}{x}$ and $g(x) = x$ which are differentiable on $(0,1)$, $g'(x) \neq 0$ for all x in $(0,1)$, $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} g(x) = 0$, and $\lim_{x\to 0+} \frac{f(x)}{g(x)} = 0$. However, $\lim_{x\to a+} \frac{f'(x)}{g'(x)}$ does not exist because $\frac{f'(x)}{g'(x)} = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$.

L' Hospital Rule II.

Let $-\infty \leq a < b \leq \infty$ and let *f*, *g* be differentiable on (a,b) such that $g'(x) \neq 0$ for all *x* in (a,b). Suppose that $\lim_{x\to a+} g(x) = \pm\infty$.

If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L$ exists in $\mathbb{R}^*$, then $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.

Proof.

Without loss of generality, suppose $\lim_{x\to a+} g(x) = \infty$. (otherwise consider $-g$)

We first consider the case that $L \in \mathbb{R}$. As before, if $\varepsilon > 0$ is given, there exists $c \in (a,b)$ such that

(1) $$L - \varepsilon < \frac{f(c) - f(\alpha)}{g(c) - g(\alpha)} < L + \varepsilon \quad \text{for } a < \alpha < c.$$

Since $\lim_{x\to a+} g(x) = \infty$, there exists $d \in (a,c)$ such that $\frac{g(\alpha) - g(c)}{g(\alpha)} > 0$ for $a < \alpha < d$. In case we can multiply (1) by $\frac{g(\alpha) - g(c)}{g(\alpha)}$ to get

$$(L-\varepsilon)\left(1 - \frac{g(c)}{g(\alpha)}\right) < \frac{f(\alpha)}{g(\alpha)} - \frac{f(c)}{g(\alpha)} < (L+\varepsilon)\left(1 - \frac{g(c)}{g(\alpha)}\right)$$

i.e.

$$(L-\varepsilon)\left(1 - \frac{g(c)}{g(\alpha)}\right) + \frac{f(c)}{g(\alpha)} < \frac{f(\alpha)}{g(\alpha)} < (L+\varepsilon)\left(1 - \frac{g(c)}{g(\alpha)}\right) + \frac{f(c)}{g(\alpha)} \quad \text{for } a < \alpha < d.$$

For any sequence (a_n) that converges to *a* with $a_n > a$ for all *n*, the first inequality suggests that $\liminf \frac{f(a_n)}{g(a_n)} \geq L$ while the second one gives $\limsup \frac{f(a_n)}{g(a_n)} \leq L$. So $\lim \frac{f(a_n)}{g(a_n)} = L$, follows that $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.

Next, we consider the case $L=\infty$ and omitted the case $L=-\infty$ since the argument is similar.

Given any $M>0$, we have

(2) $$\frac{f(c)-f(\alpha)}{g(c)-g(\alpha)}>M \quad \text{for } a<\alpha<c.$$

Again, there exists $d\in(a,c)$ such that $\frac{g(\alpha)-g(c)}{g(\alpha)}>0$ for $a<\alpha<d$. We multiply (2) by $\frac{g(\alpha)-g(c)}{g(\alpha)}$ and obtain

$$\frac{f(\alpha)}{g(\alpha)}-\frac{f(c)}{g(\alpha)}>M\left(1-\frac{g(c)}{g(\alpha)}\right)$$

i.e.

$$\frac{f(\alpha)}{g(\alpha)}>M\left(1-\frac{g(c)}{g(\alpha)}\right)+\frac{f(c)}{g(\alpha)} \quad \text{for } a<\alpha<d.$$

For any sequence (a_n) that converges to *a* from right, $\liminf\frac{f(a_n)}{g(a_n)}\geq M$. Since $M>0$ is arbitrary, $\lim\frac{f(a_n)}{g(a_n)}=\infty$ and follows that $\lim_{x\to a+}\frac{f(x)}{g(x)}=\infty$.

- As before, the converse of L' Hospital Rule II is not true. To see this, we simply consider $f(x)=\sin\frac{1}{x}$ and $g(x)=\frac{1}{x}$ on the interval $(0,1)$.