Notes on the Fundamental Theorem of Calculus

First Form.
If $F:[a,b]\to\mathbb{R}$ is continuous, $f \triangleq F'$ exists and Riemann integrable on [*a*, *b*], then

$$\int_a^b f = F(b)-F(a).$$

First Form. (Generalized Version)
If $F:[a,b]\to\mathbb{R}$ is continuous and *f* is Riemann integrable on [*a*, *b*] such that $F'(x)=f(x)$ on [*a*, *b*] except for finite number of points, then

$$\int_a^b f = F(b)-F(a).$$

- *F* may not be differentiable at every point on [*a*, *b*].

Second Form.
If *f* is integrable on [*a*, *b*] and continuous at $c\in[a,b]$, then the function $F(x)\triangleq\int_a^x f$ is differentiable at *c* and

$$F'(c)=f(c).$$

Substitution Formula.
Suppose $\varphi:[a,b]\to\mathbb{R}$ has a continuous derivative on [*a*, *b*] and $f:I\to\mathbb{R}$ is continous on an interval *I* containing $\varphi([a,b])$. Then

$$\int_a^b f\left(\varphi(t)\right)\varphi'(t)dt=\int_{\varphi(a)}^{\varphi(b)} f(x)dx.$$

Lebesgue's Integrability Criterion.
A bounded real-valued function on [*a*, *b*] is Riemann integrable if and only if it is continuous almost everywhere on [*a*, *b*].

Composition Theorem.
Let $f\in\Re[a,b]$ and $f([a,b])\subseteq[c,d]$. If $\varphi:[c,d]\to\mathbb{R}$ is continuous, then $\varphi\circ f\in\Re[a,b]$.

Integration by Parts.
Let *F*, *G* be differentiable on [*a*, *b*] and $f\triangleq F'$, $g\triangleq G'$ are Riemann integrable on [*a*, *b*], then

$$\int_a^b Fg = FG\Big|_a^b-\int_a^b fG.$$

Taylor's Theorem.
Let $f:[a,b]\to\mathbb{R}$. If $f^{(n+1)}$ exists and Riemann integrable on [*a*, *b*], then

$$f(b)=f(a)+\frac{f'(a)}{1!}(b-a)+\ldots+\frac{f^{(n)}(a)}{n!}(b-a)^n+R_n,$$

where $R_n=\dfrac{1}{n!}\displaystyle\int_a^b f^{(n+1)}(t)\cdot(b-t)^n\,dt$.

- If $f^{(n+1)}$ exists on (*a*, *b*) but we don't know it is integrable or not, then we can use $R_n=\dfrac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$ as remainder term where $c\in(a,b)$.