# Notes on MAT2050E (Elementary Analysis)
# Continuity of Functions

**Definition.**

We say that a function $f: A \to \mathbb{R}$ (where $A \subseteq \mathbb{R}$) is **continuous at** point $c \in A$ if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that if *x* is any point of *A* satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

- Following the definition, if *c* is an isolated point of *A* then *f* is automatically continuous at point *c*.

An equivalent definition of continuity can be formulated very nicely in terms of neighborhoods as follows:

**Equivalent definition.**

We say that a function $f: A \to \mathbb{R}$ (where $A \subseteq \mathbb{R}$) is **continuous at** point $c \in A$ if given any $\varepsilon -$neighborhood $V_\varepsilon(f(c))$ of $f(c)$ there exists a $\delta -$neighborhood $V_\delta(c)$ of *c* such that $f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))$.

**Sequential Criterion for Continuity.**

A function $f: A \to \mathbb{R}$ (where $A \subseteq \mathbb{R}$) is **continuous at** point $c \in A$ if and only if for every sequence $(x_n)$ in *A* that converges to *c*, the sequence $(f(x_n))$ converges to $f(c)$.

- We say that *f* is **continuous on the set B** if *f* is continuous at every point of *B*.

**Theorem.**

Let $A \subseteq \mathbb{R}$, *f* and *g* be functions on *A* to $\mathbb{R}$, and $b \in \mathbb{R}$. Suppose that $c \in A$ and that *f* and *g* are continuous at *c*.

(a) Then $f + g$, $f - g$, $fg$ and $bf$ are continuous at *c*.
(b) If $g(c) \neq 0$, then the quotient $f / g$ is continuous at *c*.

- For (b), we only need to assume $g(c) \neq 0$ since the continuity of *g* guarantees $g(x) \neq 0$ in some neighborhood of *c*. (It should be noted that the function $f / g$ may not be well-defined on *A*.)

- If *f* and *g* are continuous on *A*, the above theorem can be applied to every point of *A* so that $f + g$, $f - g$, $fg$ and $bf$ are continuous on *A*. Moreover, if $g(x) \neq 0$ for all $x \in A$, then $f / g$ is continuous on *A*.

**Theorem.**

Let $A \subseteq \mathbb{R}$. If $f : A \to \mathbb{R}$ and $g : f(A) \to \mathbb{R}$ such that *f* is continuous at $c \in A$ and *g* is continuous at $f(c) \in f(A)$, then the composition $g \circ f : A \to \mathbb{R}$ is continuous at *c*.

**Definition.**

A function $f : A \to \mathbb{R}$ is said to be **bounded on** *A* if there exists a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in A$.

**Boundedness Theorem.**

Let $I = [a,b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on *I*. Then *f* is bounded on *I*.

**Maximum-Minimum Theorem.**

Let $I = [a,b]$ and let $f : I \to \mathbb{R}$ be continuous on *I*. Then *f* has an absolute maximum and an absolute minimum on *I*.

**Location of Roots Theorem.**

Let $I = [a,b]$ and let $f : I \to \mathbb{R}$ be continuous on *I*.
If $f(a) f(b) < 0$, then there exists a number $c \in (a,b)$ such that $f(c) = 0$.

**Bolzanos' Intermediate Value Theorem.**

Let $I = [a,b]$ and let $f : I \to \mathbb{R}$ be continuous on *I*. If $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$ or $f(a) > k > f(b)$, then there exists a number $c \in (a,b)$ such that $f(c) = k$.

- The theorem follows that if $\inf f(I) < k < \sup f(I)$, then there exists $c \in I$ such that $f(c) = k$. This is true also for open intervals.

- If *I* is a closed bounded interval and $f : I \to \mathbb{R}$ is continuous on *I*, then the set $f(I)$ is also a closed bounded interval.

- If *I* is an interval (which may be open or unbounded) and $f : I \to \mathbb{R}$ is continuous on *I*, then the set $f(I)$ is also an interval. This is called the "Preservation of Intervals Theorem".

- A continuous real-valued function defined on an interval is injective (one to one) if and only if it is strictly monotone.

**Continuous Inverse Theorem.**

Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on *I*. Then the function *g* inverse to *f* is strictly monotone and continuous on $f(I)$.