Exercises on Inequality

Problem 1.

Let a, b and c be positive real numbers such that $abc = 1$.

Prove that $\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$.

Problem 2.

Let n be a positive integer.

Prove that the average of the numbers $\sqrt{1},\sqrt{2},\sqrt{3},\cdots,\sqrt{n}$ exceeds $\frac{2}{3}\sqrt{n}$.

Problem 3.

Let $a_1,a_2,\cdots,a_n$ be positive real numbers such that $a_1+a_2+\cdots+a_n<1$. Prove that

$$\frac{a_1a_2\cdots a_n\left[1-(a_1+a_2+\cdots+a_n)\right]}{(a_1+a_2+\cdots+a_n)(1-a_1)(1-a_2)\cdots(1-a_n)}\leq\frac{1}{n^{n+1}}.$$

Problem 4.

Let x, y and z be positive real numbers such that $x+y+z\geq 3$. Prove that

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)}\geq\frac{3}{4}.$$

Problem 5.

Let a, b, c be positive real numbers. Prove that

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)\geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right).$$

Problem 6.

Let n be a fixed integer, with $n\geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1\leq i<j\leq n}x_ix_j(x_i^2+x_j^2)\leq C\Big(\sum_{1\leq i\leq n}x_i\Big)^4$$

holds for all real numbers $x_1,\cdots,x_n\geq 0$.

(b) For this constant C, determine when equality holds.