A List of Important Inequalities

1. **(AM-GM-HM Inequality)** For $a_1, a_2, \cdots, a_n > 0$,

$$AM = \frac{a_1 + a_2 + \cdots + a_n}{n} \geq GM = \sqrt[n]{a_1 a_2 \cdots a_n} \geq HM = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

Either equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Remark：For **AM-GM Inequality**, the condition for the a_i's may be relaxed to $a_1, a_2, \cdots, a_n \geq 0$.

2. **(Cauchy-Schwarz Inequality)** For real numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$,

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2).$$

Equality holds if and only if $a_i b_j = a_j b_i$ for $1 \leq i, j \leq n$.

3. **(Chebysev's Inequality)** If $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, then

$$\begin{aligned} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n &\geq \frac{(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)}{n} \\ &\geq a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1 \end{aligned}.$$

Either equality holds if and only if $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

4. **(Triangular Inequality)** For real numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$,

$$\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} + \sqrt{b_1^2 + b_2^2 + \ldots b_n^2} \geq \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + \ldots + (a_n + b_n)^2}.$$

5. **(Jensen's Inequality)** If a function *f* is convex on an interval *I* and $x_1, x_2, \cdots, x_n \in I$, then

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

For strictly convex functions, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

More generally, if $0 < t_1, t_2, \cdots, t_n < 1$ and $t_1 + t_2 + \cdots + t_n = 1$, then

$$f(t_1 x_1 + t_2 x_2 + \cdots + t_n x_n) \leq t_1 f(x_1) + t_2 f(x_2) + \cdots + t_n f(x_n),$$

with the same equality condition. (similar inequality holds for concave functions)

6. **(Majorization Inequality)** If a function f is convex on an interval I, $x_i, y_i \in I$, $(x_1, x_2, \cdots, x_n) \succ (y_1, y_2, \cdots, y_n)$, then

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq f(y_1) + f(y_2) + \cdots + f(y_n).$$

For strictly convex functions, equality holds if and only if $x_i = y_i$ for $i = 1, 2, \cdots, n$. (similar inequality holds for concave functions with the same equality condition)

Remarks：

(i.) If $f''(x) \geq 0$ on (a,b), then f is convex on (a,b).

(i.) If $f''(x) > 0$ on (a,b), then f is strictly convex on (a,b).

(ii.) The following functions are strictly convex on the given intervals:

$$x^p \text{ on } [0,\infty) \text{ for } p > 1,$$

$$x^p \text{ on } (0,\infty) \text{ for } p < 0,$$

$$a^x \text{ on } (-\infty,\infty) \text{ for } a > 1.$$

(iii.) The following functions are strictly concave on the given intervals:

$$x^p \text{ on } [0,\infty) \text{ for } 0 < p < 1,$$

$$\log_a x \text{ on } (0,\infty) \text{ for } a > 1.$$

(iv.) If $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ satisfy the conditions $\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$ for $1 \leq k \leq n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then we say $(x_1, x_2, \cdots, x_n)$ majorizes $(y_1, y_2, \cdots, y_n)$ and write $(x_1, x_2, \cdots, x_n) \succ (y_1, y_2, \cdots, y_n)$.

7. **(Muirhead's Theorem)** If $s = (s_1, s_2, \cdots, s_n)$ and $t = (t_1, t_2, \cdots, t_n)$ are sequences of nonnegative real numbers such that s majorizes t, then

$$\sum_{sym} x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n} \geq \sum_{sym} x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}$$

for all nonnegative $x_1, x_2, \cdots, x_n$. Conversely, if this inequality holds for all nonnegative $x_1, x_2, \cdots, x_n$, then s majorizes t.

8. **(Schur)** Let x, y, z be nonnegative real numbers. Then for any $r > 0$,

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0.$$

Equality holds if and only if $x = y = z$, or if two of x, y, z are equal and the third is zero.

9. **(Permutation Inequality)** If $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, then

$$\begin{aligned} & a_1b_1 + a_2b_2 + \cdots + a_nb_n \quad (\textit{ordered sum}) \\ \ge\ & a_1b_{r1} + a_2b_{r2} + \cdots + a_nb_{rn} \quad (\textit{mixed sum}) \quad , \\ \ge\ & a_1b_n + a_2b_{n-1} + \cdots + a_nb_1 \quad (\textit{reverse sum}) \end{aligned}$$

where $(r_1, r_2, \cdots, r_n)$ is a permutation of $(1, 2, \cdots, n)$. Either equality holds if and only if $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

10. **(Maclaurin's Symmetric Mean Inequality)** For $a_1, a_2, \cdots, a_n > 0$,

$$AM = P_1 \ge P_2^{1/2} \ge \cdots \ge P_n^{1/n} = GM ,$$

where $P_j = (\sum a_{i1}a_{i2}\cdots a_{ij}) \div C_j^n$ is the average of all possible products of $a_1, a_2, \cdots, a_n$ taken j at a time. Moreover, any equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

11. **(Power Mean Inequality)** For $a_1, a_2, \cdots, a_n > 0$ and $s < t$,

$$M_s = \left(\frac{a_1^s + a_2^s + \cdots + a_n^s}{n}\right)^{1/s} \le M_t = \left(\frac{a_1^t + a_2^t + \cdots + a_n^t}{n}\right)^{1/t}.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

12. **(Bernoulli's Inequality)** For real number $x > -1$,

$$(1+x)^\alpha \begin{cases} \ge 1 + \alpha x & \text{if } \alpha < 0 \text{ or } \alpha > 1 \\ \le 1 + \alpha x & \text{if } 0 < \alpha < 1 \end{cases}.$$

Equality holds if and only if $x = 0$.

13. **(微微對偶不等式)** If $0 \le a_{i1} \le a_{i2} \le \cdots \le a_{in}$ $(i = 1, 2, \cdots, m)$ and $a'_{i1}, a'_{i2}, \cdots, a'_{in}$ is a permutation of $a_{i1}, a_{i2}, \cdots, a_{in}$, then

$$\begin{aligned} & (a_{11}a_{21}\cdots a_{m1}) + (a_{12}a_{22}\cdots a_{m2}) + \cdots + (a_{1n}a_{2n}\cdots a_{mn}) \\ \ge\ & (a'_{11}a'_{21}\cdots a'_{m1}) + (a'_{12}a'_{22}\cdots a'_{m2}) + \cdots + (a'_{1n}a'_{2n}\cdots a'_{mn}) \end{aligned}$$

and

$$\begin{aligned} & (a_{11} + a_{21} + \cdots + a_{m1})(a_{12} + a_{22} + \cdots + a_{m2})\cdots(a_{1n} + a_{2n} + \cdots + a_{mn}) \\ \le\ & (a'_{11} + a'_{21} + \cdots + a'_{m1})(a'_{12} + a'_{22} + \cdots + a'_{m2})\cdots(a'_{1n} + a'_{2n} + \cdots + a'_{mn}) \end{aligned}.$$