



**Inequalities (Unit 1)**

1. By the AM-GM inequality, we have  $\frac{1+a_i}{2} \geq \sqrt{1 \cdot a_i}$ , i.e.  $1+a_i \geq 2\sqrt{a_i}$  for all  $i$ . Hence

$$\begin{aligned} 2^n &= (1+a_1)(1+a_2)\cdots(1+a_n) \\ &\geq (2\sqrt{a_1})(2\sqrt{a_2})\cdots(2\sqrt{a_n}) \\ &= 2^n \sqrt{a_1 a_2 \cdots a_n} \end{aligned}$$

Dividing both side by  $2^n$ , we have  $1 \geq \sqrt{a_1 a_2 \cdots a_n}$ , so that  $a_1 a_2 \cdots a_n \leq 1$ .

2. By the AM-GM inequality, we have  $\frac{1}{2} \left( \frac{a_1^2}{a_2} + a_2 \right) \geq \sqrt{\frac{a_1^2}{a_2} \cdot a_2}$ , i.e.  $\frac{a_1^2}{a_2} \geq 2a_1 - a_2$ .

Similarly, we have  $\frac{a_1^2}{a_2} \geq 2a_1 - a_2$ ,  $\frac{a_2^2}{a_3} \geq 2a_2 - a_3$  and so on. Hence

$$\begin{aligned} \frac{a_1^2}{a_2} + \frac{a_2^2}{a_3} + \cdots + \frac{a_n^2}{a_1} &\geq (2a_1 - a_2) + (2a_2 - a_3) + \cdots + (2a_n - a_1) \\ &= (2a_1 + 2a_2 + \cdots + 2a_n) - (a_2 + a_3 + \cdots + a_n + a_1) \\ &= a_1 + a_2 + \cdots + a_n \end{aligned}$$

Alternative Solution

Without loss of generality, assume  $a_1 \geq a_2 \geq \cdots \geq a_n$ . Then we have

$$a_1^2 \geq a_2^2 \geq \cdots \geq a_n^2 \quad \text{and} \quad \frac{1}{a_1} \leq \frac{1}{a_2} \leq \cdots \leq \frac{1}{a_n}.$$

Using the fact that Random Sum  $\geq$  Reverse Sum, we have

$$\frac{a_1^2}{a_2} + \frac{a_2^2}{a_3} + \cdots + \frac{a_n^2}{a_1} \geq \frac{a_1^2}{a_1} + \frac{a_2^2}{a_2} + \cdots + \frac{a_n^2}{a_n} = a_1 + a_2 + \cdots + a_n.$$

3. Let  $x=1-a$ ,  $y=1-b$  and  $z=1-c$ .

Then  $a+b+c=2$  implies  $a=2-b-c=2-(1-y)-(1-z)=y+z$ .

Similarly, we have  $b=z+x$  and  $c=x+y$ .

Hence the original inequality becomes  $\frac{(x+y)(y+z)(z+x)}{xyz} \geq 8$ , or  $(x+y)(y+z)(z+x) \geq 8xyz$ .

By the AM-GM inequality, we have  $\frac{x+y}{2} \geq \sqrt{xy}$ , i.e.  $x+y \geq 2\sqrt{xy}$ .

Similarly,  $y+z \geq 2\sqrt{yz}$  and  $z+x \geq 2\sqrt{zx}$ .

Consequently,  $(x+y)(y+z)(z+x) \geq (2\sqrt{xy})(2\sqrt{yz})(2\sqrt{zx}) = 8xyz$ , completing the proof.

4. Without loss of generality, assume  $a \geq b \geq c$ . Then

$$\frac{1}{b+c} \geq \frac{1}{a+c} \geq \frac{1}{a+b}.$$

Using the fact that Direct Sum  $\geq$  Random Sum, we have

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{b}{b+c} + \frac{c}{a+c} + \frac{a}{a+b}$$

Taking another random sum, we have

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{c}{b+c} + \frac{a}{a+c} + \frac{b}{a+b}.$$

Adding the above two inequalities, we have

$$2\left(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}\right) \geq \frac{b+c}{b+c} + \frac{a+c}{a+c} + \frac{a+b}{a+b} = 3,$$

so that  $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$ .

### Alternative Solution

By the Cauchy-Schwarz inequality, we have

$$\left(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}\right)[a(b+c) + b(c+a) + c(a+b)] \geq (a+b+c)^2.$$

Hence it suffices to prove that

$$\frac{(a+b+c)^2}{a(b+c) + b(c+a) + c(a+b)} \geq \frac{3}{2}.$$

By the AM-GM inequality,

$$\begin{aligned} 2(a^2 + b^2 + c^2) &= 2a^2 + 2b^2 + 2c^2 + 4ab + 4bc + 4ca \\ &= (a^2 + b^2) + (b^2 + c^2) + (c^2 + a^2) + 4ab + 4bc + 4ca \\ &\geq 2ab + 2bc + 2ca + 4ab + 4bc + 4ca \\ &= 3[a(b+c) + b(c+a) + c(a+b)] \end{aligned}$$

so that the desired inequality follows.

5. Let  $b_1 < b_2 < \dots < b_n$  be a permutation of  $a_1, a_2, \dots, a_n$  in ascending order.

Since  $a_1, a_2, \dots, a_n$  are distinct positive integers, we have  $b_i \geq i$  for all  $i$ .

Using the fact that  $1 \geq \frac{1}{2^2} \geq \dots \geq \frac{1}{n^2}$  and that Random Sum  $\geq$  Reverse Sum, we have

$$\begin{aligned} a_1 + \frac{a_2}{2^2} + \dots + \frac{a_n}{n^2} &\geq b_1 + \frac{b_2}{2^2} + \dots + \frac{b_n}{n^2} \\ &\geq 1 + \frac{2}{2^2} + \dots + \frac{n}{n^2} \\ &= 1 + \frac{1}{2} + \dots + \frac{1}{n} \end{aligned}$$

## Inequalities (Unit 2)

1. Setting  $z = x + y$  and taking cube root on both sides, the original inequality becomes

$$x^2 + y^2 + (x + y)^2 \geq 3 \cdot \sqrt[3]{2x^{\frac{2}{3}}y^{\frac{2}{3}}(x + y)^{\frac{2}{3}}}.$$

Now, using the AM-GM inequality twice, we have

$$\begin{aligned}x^2 + y^2 + (x + y)^2 &= \frac{x^2 + y^2}{2} + \frac{x^2 + y^2}{2} + (x + y)^2 \\&\geq \frac{x^2 + y^2}{2} + \frac{2xy}{2} + (x + y)^2 \\&= \frac{3}{2}(x + y)^2 \\&= \frac{3}{2}(x + y)^{\frac{4}{3}}(x + y)^{\frac{2}{3}} \\&\geq \frac{3}{2}(2\sqrt{xy})^{\frac{4}{3}}(x + y)^{\frac{2}{3}} \\&= 3 \cdot \sqrt[3]{2x^{\frac{2}{3}}y^{\frac{2}{3}}(x + y)^{\frac{2}{3}}}\end{aligned}$$

Hence the original inequality is proved.

2. Without loss of generality, assume  $x \geq y \geq z$ . Let  $z = \frac{1}{3} - k$ . Then  $0 \leq k \leq \frac{1}{3}$ .

Using the facts that  $x + y = \frac{2}{3} + k$  and  $xy \leq \left(\frac{x + y}{2}\right)^2 = \left(\frac{1}{3} + \frac{k}{2}\right)^2$ , we get

$$\begin{aligned}xy + yz + zx - 3xyz &\leq z(x + y) + \left(\frac{x + y}{2}\right)^2(1 - 3z) \\&= \left(\frac{1}{3} - k\right)\left(\frac{2}{3} + k\right) + \left(\frac{1}{3} + \frac{k}{2}\right)^2(3k) \\&= \frac{2}{9} + \frac{3}{4}k^3 \\&\leq \frac{2}{9} + \frac{3}{4}\left(\frac{1}{3}\right)^3 \\&\leq \frac{1}{4}\end{aligned}$$

### Alternative Solution

Note that

$$\begin{aligned}
xy + yz + zx - 3xyz &= xy(1-z) + yz(1-x) + xz(1-y) \\
&= xy(x+y) + yz(y+z) + xz(z+x) \\
&= x^2(y+z) + y^2(x+z) + z^2(y+x) \\
&= x^2(1-x) + y^2(1-y) + z^2(1-z)
\end{aligned}$$

Since  $x^2(1-x) - \frac{x}{4} = -\frac{x}{4}(1-2x)^2 \leq 0$ , we have  $x^2(1-x) \leq \frac{x}{4}$ .

Similarly,  $y^2(1-y) \leq \frac{y}{4}$  and  $z^2(1-z) \leq \frac{z}{4}$ . Consequently,

$$xy + yz + zx - 3xyz = x^2(1-x) + y^2(1-y) + z^2(1-z) \leq \frac{x}{4} + \frac{y}{4} + \frac{z}{4} = \frac{1}{4}.$$

3. After some trial, we find that equality holds when the two triangles on the left hand side are similar, i.e. when

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'}.$$

This is clearly the equality condition for the Cauchy-Schwarz inequality. Therefore, we attempt to use the Cauchy-Schwarz inequality to solve the problem.

Now we must express the area of a triangle in terms of its side lengths. Clearly, we should use the Heron's formula, which states that the area of a triangle with side lengths  $a, b, c$  is

$$\sqrt{s(s-a)(s-b)(s-c)},$$

where  $s = \frac{a+b+c}{2}$ . Hence the original inequality becomes

$$\begin{aligned}
&\sqrt[4]{s(s-x)(s-y)(s-z)} + \sqrt[4]{s'(s'-x')(s'-y')(s'-z')} \\
&\leq \sqrt[4]{(s+s')(s+s'-x-x')(s+s'-y-y')(s+s'-z-z')}
\end{aligned}$$

with  $s = \frac{x+y+z}{2}$  and  $s' = \frac{x'+y'+z'}{2}$ . Now, using the Cauchy-Schwarz inequality twice, we have

$$\begin{aligned}
&\sqrt[4]{s(s-x)(s-y)(s-z)} + \sqrt[4]{s'(s'-x')(s'-y')(s'-z')} \\
&\leq \sqrt{\left[\sqrt{s(s-x)} + \sqrt{s'(s'-x')}\right] \cdot \left[\sqrt{(s-y)(s-z)} + \sqrt{(s'-y')(s'-z')}\right]} \\
&\leq \sqrt{\sqrt{(s+s')(s-x+s'-x')} \cdot \sqrt{(s-y+s'-y')(s-z+s'-z')}} \\
&= \sqrt[4]{(s+s')(s+s'-x-x')(s+s'-y-y')(s+s'-z-z')}
\end{aligned}$$

and so the original inequality is proved.

4. Let  $a_{2004} = 1 - a_1 - a_2 - \dots - a_{2003}$ . Then  $a_1 + a_2 + \dots + a_{2004} = 1$  and

$$\frac{a_1 a_2 \cdots a_{2003} (1 - a_1 - a_2 - \dots - a_{2003})}{(a_1 + a_2 + \dots + a_{2003})(1 - a_1)(1 - a_2) \cdots (1 - a_{2003})} = \frac{a_1 a_2 \cdots a_{2004}}{(1 - a_1)(1 - a_2) \cdots (1 - a_{2004})}.$$

By the AM-GM inequality,

$$\begin{aligned} & (1 - a_1)(1 - a_2) \cdots (1 - a_{2004}) \\ &= (a_2 + a_3 + \dots + a_{2004})(a_1 + a_3 + \dots + a_{2004}) \cdots (a_1 + a_2 + \dots + a_{2003}) \\ &\geq \left(2003 \cdot \sqrt[2003]{a_2 a_3 \cdots a_{2004}}\right) \left(2003 \cdot \sqrt[2003]{a_1 a_3 \cdots a_{2004}}\right) \cdots \left(2003 \cdot \sqrt[2003]{a_1 a_2 \cdots a_{2003}}\right) \\ &= 2003^{2004} a_1 a_2 \cdots a_{2004} \end{aligned}$$

Hence we have  $\frac{a_1 a_2 \cdots a_{2004}}{(1 - a_1)(1 - a_2) \cdots (1 - a_{2004})} \leq \frac{1}{2003^{2004}}$ .

Furthermore, equality holds when  $a_1 = a_2 = \dots = a_{2004} = \frac{1}{2004}$ .

Therefore, the answer is  $\frac{1}{2003^{2004}}$ .

5. By the Cauchy-Schwarz inequality,  $\sum_{k=1}^n \left( \frac{a_k^2}{a_k + b_k} \right) \cdot \sum_{k=1}^n (a_k + b_k) \geq \left( \sum_{k=1}^n a_k \right)^2$ .

Hence  $\sum_{k=1}^n \left( \frac{a_k^2}{a_k + b_k} \right) \geq \frac{\left( \sum_{k=1}^n a_k \right)^2}{\sum_{k=1}^n (a_k + b_k)} = \frac{\left( \sum_{k=1}^n a_k \right)^2}{2 \cdot \sum_{k=1}^n a_k} = \sum_{k=1}^n \left( \frac{a_k}{2} \right)$ .

### Alternative Solution

For real numbers  $a$  and  $b$ , we have  $(a + b)^2 \geq (2\sqrt{ab})^2 = 4ab$ , so  $\frac{ab}{a + b} \leq \frac{a + b}{4}$ . Hence

$$\begin{aligned}
\sum_{k=1}^n \left( \frac{a_k^2}{a_k + b_k} \right) &= \sum_{k=1}^n \left( \frac{a_k^2 + a_k b_k - a_k b_k}{a_k + b_k} \right) \\
&= \sum_{k=1}^n a_k - \sum_{k=1}^n \left( \frac{a_k b_k}{a_k + b_k} \right) \\
&\geq \sum_{k=1}^n a_k - \sum_{k=1}^n \left( \frac{a_k + b_k}{4} \right) \\
&= \sum_{k=1}^n a_k - \sum_{k=1}^n \left( \frac{2a_k}{4} \right) \\
&= \sum_{k=1}^n \left( \frac{a_k}{2} \right)
\end{aligned}$$

6. Let  $x = a + b - c$ ,  $y = b + c - a$  and  $z = c + a - b$ . Then the original inequality becomes

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leq \sqrt{\frac{x+y}{2}} + \sqrt{\frac{y+z}{2}} + \sqrt{\frac{z+x}{2}}.$$

By the AM-GM inequality, we have

$$\left( \frac{\sqrt{x} + \sqrt{y}}{2} \right)^2 = \frac{x + y + 2\sqrt{xy}}{4} = \frac{x+y}{4} + \frac{\sqrt{xy}}{2} \leq \frac{x+y}{4} + \frac{x+y}{4} = \frac{x+y}{2}$$

and hence  $\frac{\sqrt{x} + \sqrt{y}}{2} \leq \sqrt{\frac{x+y}{2}}$ . Similarly, we have

$$\frac{\sqrt{y} + \sqrt{z}}{2} \leq \sqrt{\frac{y+z}{2}} \quad \text{and} \quad \frac{\sqrt{z} + \sqrt{x}}{2} \leq \sqrt{\frac{z+x}{2}}.$$

Adding these three inequalities, we have

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leq \sqrt{\frac{x+y}{2}} + \sqrt{\frac{y+z}{2}} + \sqrt{\frac{z+x}{2}},$$

thereby proving the original inequality.

Finally, equality in the above application of AM-GM inequality occurs if  $\sqrt{x} = \sqrt{y}$ , i.e.  $x = y$ . Similarly we must have  $y = z$  and  $z = x$ . If  $x = y$ , then  $a + b - c = b + c - a$ , hence  $2a = 2c$  and  $a = c$ . Similarly, we must have  $a = b$  and  $b = c$ . That is, equality holds if and only if  $a = b = c$ .

7. By the Cauchy-Schwarz inequality, we have

$$(x^2 + y^2 + z^2)(1^2 + 1^2 + 1^2) \geq (x + y + z)^2$$

which gives  $x + y + z \leq \sqrt{3(x^2 + y^2 + z^2)}$ . On the other hand, the AM-GM inequality asserts that

$$xy + yz + zx \geq 3(xyz)^{\frac{2}{3}} \text{ and } \sqrt{x^2 + y^2 + z^2} \geq \sqrt{3}(xyz)^{\frac{1}{3}}.$$

Consequently, we have

$$\begin{aligned} \frac{xyz(x + y + z + \sqrt{x^2 + y^2 + z^2})}{(x^2 + y^2 + z^2)(xy + yz + zx)} &\leq \frac{xyz(\sqrt{3(x^2 + y^2 + z^2)} + \sqrt{x^2 + y^2 + z^2})}{(x^2 + y^2 + z^2)(xy + yz + zx)} \\ &= \frac{xyz(\sqrt{3} + 1)(\sqrt{x^2 + y^2 + z^2})}{(x^2 + y^2 + z^2)(xy + yz + zx)} \\ &= \frac{(\sqrt{3} + 1)xyz}{(\sqrt{x^2 + y^2 + z^2})(xy + yz + zx)} \\ &\leq \frac{(\sqrt{3} + 1)xyz}{\sqrt{3}(xyz)^{\frac{1}{3}} \cdot 3(xyz)^{\frac{2}{3}}} \\ &= \frac{\sqrt{3} + 1}{3\sqrt{3}} \\ &= \frac{3 + \sqrt{3}}{9} \end{aligned}$$

8. By the rearrangement inequality, we have  $a^3 + b^3 \geq a^2b + ab^2 = ab(a + b)$ .

Similarly, we have  $b^3 + c^3 \geq bc(b + c)$  and  $c^3 + a^3 \geq ca(c + a)$ . Consequently,

$$\begin{aligned} &\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \\ &\leq \frac{1}{ab(a + b) + abc} + \frac{1}{bc(b + c) + abc} + \frac{1}{ca(c + a) + abc} \\ &= \frac{1}{ab(a + b + c)} + \frac{1}{bc(a + b + c)} + \frac{1}{ca(a + b + c)} \\ &= \frac{1}{abc} \end{aligned}$$

9. Let  $a - 1 = x^2$ ,  $b - 1 = y^2$  and  $c - 1 = z^2$  for some positive non-negative  $x, y, z$ . Then the original inequality becomes

$$x + y + z \leq \sqrt{(z^2 + 1)[(x^2 + 1)(y^2 + 1) + 1]}.$$

By the Cauchy-Schwarz inequality, we have

$$x + y = x \cdot 1 + 1 \cdot y \leq \sqrt{(x^2 + 1)(y^2 + 1)}.$$

Similarly, we have

$$x + y + z \leq \sqrt{[(x + y)^2 + 1](z^2 + 1)} \leq \sqrt{[(x^2 + 1)(y^2 + 1) + 1](z^2 + 1)}$$

and proof is complete.

10. (a) When  $n = 2$  and  $x_1 = x_2 = 1$ ,  $C \geq \frac{1(1)(1^2 + 1^2)}{(1 + 1)^4} = \frac{1}{8}$ .

On the other hand, when  $C = \frac{1}{8}$ , the inequality holds for all real numbers  $x_1, \dots, x_n \geq 0$  since

$$\begin{aligned} \frac{1}{8} \left( \sum_{1 \leq i \leq n} x_i \right)^4 &= \frac{1}{8} \left[ \left( \sum_{1 \leq i \leq n} x_i \right)^2 \right]^2 \\ &= \frac{1}{8} \left[ \left( \sum_{1 \leq i \leq n} x_i^2 \right) + 2 \left( \sum_{1 \leq i < j \leq n} x_i x_j \right) \right]^2 \\ &\geq \frac{1}{8} \left[ 2 \sqrt{2 \left( \sum_{1 \leq i < j \leq n} x_i x_j \right) \left( \sum_{1 \leq i \leq n} x_i^2 \right)} \right]^2 && \text{(AM-GM inequality)} \\ &= \left( \sum_{1 \leq i < j \leq n} x_i x_j \right) \left( \sum_{1 \leq i \leq n} x_i^2 \right) \\ &= \sum_{1 \leq i < j \leq n} x_i x_j (x_1^2 + x_2^2 + \dots + x_n^2) \\ &\geq \sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \end{aligned}$$

Hence the required least constant  $C$  is  $\frac{1}{8}$ .

(b) Consider the term with  $i = 1$  and  $j = 2$  in the last two expressions in (a).

$$\text{We have } x_1 x_2 (x_1^2 + x_2^2 + \dots + x_n^2) \geq x_1 x_2 (x_1^2 + x_2^2).$$

This equality holds if and only if  $x_3 = x_4 = \dots = x_n$ .

As the choice of  $i$  and  $j$  is arbitrary, if any  $(n - 2)$  of the  $x_i$ 's are zero, then equality in the last inequality holds, and vice versa

When  $(n - 2)$  of the  $x_i$ 's are zero, the inequality is reduced to the case of  $n = 2$ .

Consider the application of AM-GM inequality in (a).

Equality holds if and only if  $x_1^2 + x_2^2 = 2x_1x_2$ , or  $(x_1 - x_2)^2 = 0$ , i.e.  $x_1 = x_2$ .

Hence equality of the original inequality holds if and only if any  $(n - 2)$  of the  $x_i$ 's are zero and the remaining two  $x_i$ 's are equal (possibly to zero).

### Inequalities (Unit 3)

1. By Heron's formula,  $T = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)}$ .

Putting this into the original inequality, the original inequality can be simplified as follows:

$$\begin{aligned} a^2 + b^2 + c^2 &\geq \sqrt{3(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \\ a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 &\geq 3[(b+c)^2 - a^2][a^2 - (b-c)^2] \\ a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 &\geq 3[2bc + (b^2 + c^2 - a^2)][2bc - (b^2 + c^2 - a^2)] \\ a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 &\geq 3[2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4] \\ 4a^4 + 4b^4 + 4c^4 &\geq 4a^2b^2 + 4b^2c^2 + 4c^2a^2 \end{aligned}$$

By the AM-GM inequality, we have

$$\begin{aligned} 4a^4 + 4b^4 + 4c^4 &= (2a^4 + 2b^4) + (2b^4 + 2c^4) + (2c^4 + 2a^4) \\ &\geq (2\sqrt{2a^4 \cdot 2b^4}) + (2\sqrt{2b^4 \cdot 2c^4}) + (2\sqrt{2c^4 \cdot 2a^4}) \\ &= 4a^2b^2 + 4b^2c^2 + 4c^2a^2 \end{aligned}$$

thereby proving the last inequality and hence the original inequality. It is clear in the application of the AM-GM inequality that equality holds if and only if  $a = b = c$ .

#### Alternative Solution

Without loss of generality, assume that the angle opposite the side  $a$  is acute. Suppose that the altitude from this vertex, whose length we denote by  $h$ , is of distances  $m$  and  $n$  from the remaining 2 vertices, with  $b = \sqrt{h^2 + m^2}$  and  $c = \sqrt{h^2 + n^2}$ . WLOG, assume that  $m \geq n$ . Then  $a = m+n$  or  $m-n$ .

For  $a = m+n$ , the original inequality becomes

$$(m+n)^2 + (h^2 + m^2) + (h^2 + n^2) \geq 4\sqrt{3} \cdot \frac{(m+n)h}{2}.$$

Rewriting this as a quadratic equality in  $h$ , we have

$$h^2 - \sqrt{3}(m+n)h + (m^2 + mn + n^2) \geq 0.$$

The discriminant of the quadratic function on the left is

$$\Delta = [\sqrt{3}(m+n)]^2 - 4(1)(m^2 + mn + n^2) = -(m-n)^2 \leq 0.$$

Since the coefficient of  $h^2$  is positive, this means  $h^2 - \sqrt{3}(m+n)h + (m^2 + mn + n^2) \leq 0$  for all  $h$ , as desired. Equality holds when  $m = n$ , which means  $b = c$ . By symmetry, we need  $a = b = c$ .

For  $a = m-n$ , the argument is the same as above, with  $n$  replaced by  $-n$ .

2. Rewrite the given inequality as  $c^2 = a^2 + b^2 - 2ab \cos 60^\circ$ .

Hence we see that  $a, b, c$  are the side lengths of a triangle where the angle opposite the side with length  $c$  is equal to  $60^\circ$ .

In a triangle, a side opposite a larger angle is longer. Since  $60^\circ = 180^\circ \div 3$ , one other angle of the triangle must be at least  $60^\circ$  and the remaining angle must be at most  $60^\circ$ . In other words, if we assume (without loss of generality) that  $a \geq b$ , then we must have  $a \geq c$  and  $b \leq c$ .

From this, we see that  $a - c$  is positive while  $b - c$  is negative, so that  $(a - c)(b - c) \leq 0$ .

3. By the Cauchy-Schwarz inequality, we have

$$\left( \frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF} \right) (BC \cdot PD + CA \cdot PE + AB \cdot PF) \geq (BC + CA + AB)^2.$$

Hence

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF} \geq \frac{(BC + CA + AB)^2}{BC \cdot PD + CA \cdot PE + AB \cdot PF}.$$

The right hand side of the above inequality is a constant, since the numerator is the square of the perimeter while the denominator is twice the area.

Equality holds if and only if

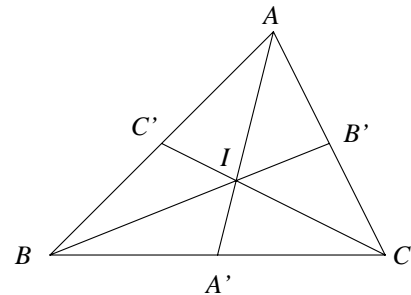
$$\frac{BC}{PD} : \frac{CA}{PE} : \frac{AB}{PF} = (BC \cdot PD) : (CA \cdot PE) : (AB \cdot PF),$$

or  $PD = PE = PF$ . In other words, the expression in the question is minimum when (and only when)  $P$  is the incentre of  $\triangle ABC$ .

4. Let  $x = \frac{AI}{AA'}$ ,  $y = \frac{BI}{BB'}$  and  $z = \frac{CI}{CC'}$ . The inequality to be proved is then  $\frac{1}{4} \leq xyz \leq \frac{8}{27}$ .

Note that

$$\begin{aligned} & x + y + z \\ &= \frac{AI}{AP} + \frac{BI}{BQ} + \frac{CI}{CR} \\ &= \frac{[ABI] + [CAI]}{[ABC]} + \frac{[BAI] + [BCI]}{[ABC]} + \frac{[CAI] + [CBI]}{[ABC]} \\ &= \frac{2([ABI] + [BCI] + [CAI])}{[ABC]} \\ &= \frac{2[ABC]}{[ABC]} \\ &= 2 \end{aligned}$$



Hence the AM-GM inequality asserts that

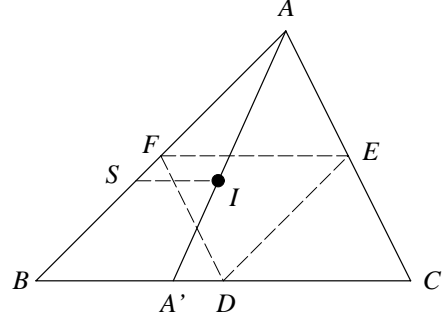
$$xyz \leq \left( \frac{x+y+z}{3} \right)^3 = \left( \frac{2}{3} \right)^3 = \frac{8}{27},$$

thereby proving the right-hand inequality.

To prove the left-hand inequality, we first make some additional observation as follows. Let  $D, E, F$  be the mid-points of  $BC, CA, AB$  respectively. We claim that  $I$  lies in  $\triangle DEF$ . Assuming the claim, we draw a line through  $I$  parallel to  $BC$  cutting  $AB$  at  $S$ . Since  $\triangle ASI \sim \triangle ABA'$ , we have

$$x = \frac{AI}{AA'} = \frac{AS}{AB} > \frac{AF}{AB} = \frac{1}{2}.$$

Similarly, we have  $y > \frac{1}{2}$  and  $z > \frac{1}{2}$ .



Now we return to the proof of the claim, namely, that  $I$  lies in  $\triangle DEF$ . Indeed, the angle bisector theorem yields

$$\frac{AI}{IA'} = \frac{AB}{BA'} \quad \text{and} \quad \frac{BA'}{A'C} = \frac{AB}{AC},$$

so that  $\frac{BA'}{BC} = \frac{AB}{AB+AC}$  and hence  $\frac{AI}{IA'} = \frac{AB+AC}{BC} > \frac{BC}{BC} = 1$  by the triangle inequality. Hence  $I$  is 'below'  $EF$  in the figure, and the same is true with respect to  $DF$  and  $DE$ , thereby establishing the claim.

Consequently, we may write

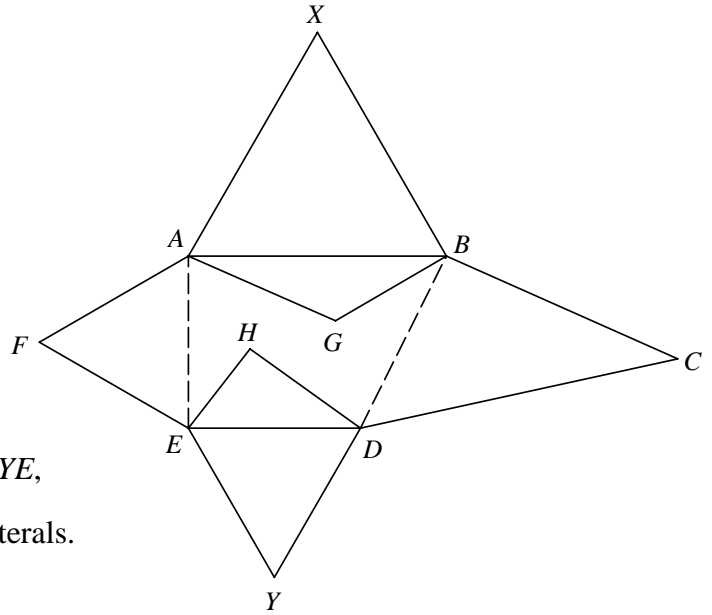
$$x = \frac{1}{2} + \alpha, \quad y = \frac{1}{2} + \beta \quad \text{and} \quad z = \frac{1}{2} + \gamma$$

for some  $\alpha, \beta, \gamma > 0$ . Then we have

$$\begin{aligned} xyz &= \left( \frac{1}{2} + \alpha \right) \left( \frac{1}{2} + \beta \right) \left( \frac{1}{2} + \gamma \right) \\ &= \frac{1}{8} + \frac{1}{4}(\alpha + \beta + \gamma) + \frac{1}{2}(\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha\beta\gamma \\ &> \frac{1}{8} + \frac{1}{4}(\alpha + \beta + \gamma) \\ &= \frac{1}{8} + \frac{1}{4} \left( \frac{1}{2} \right) \\ &= \frac{1}{4} \end{aligned}$$

and hence proving the left-hand inequality.

5. Let  $AB = BC = CD = a$  and  $DE = EF = FA = b$ . As shown in the figure, construct equilateral triangles  $ABX$  and  $DEY$ . Since  $\angle AXB = \angle DYE = 60^\circ$ ,  $AX = XB = BD = a$  and  $DY = YE = EA = b$ , the two hexagons  $ABCDEF$  and  $AXBDEY$  are congruent and so  $CF = XY$ .



Since

$$\angle AXB + \angle AGB = 180^\circ = \angle DHE + \angle DYE,$$

$AXBG$  and  $DYEH$  are cyclic quadrilaterals.

Hence by the Ptolemy's theorem,

$$AB \cdot XG = AX \cdot BG + XB \cdot AG,$$

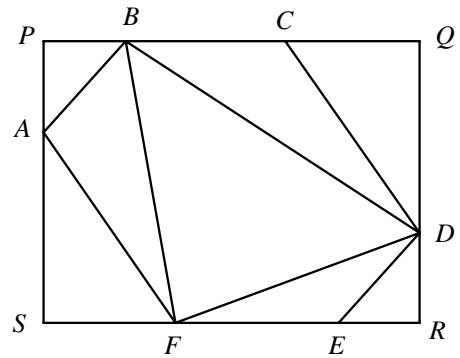
which is equivalent to  $aXG = aBG + aAG$ , or

$$XG = BG + AG.$$

Similarly, we have  $YH = DH + EH$  and hence

$$AG + GB + GH + DH + HE = XG + GH + YH \geq XY = EF.$$

6. As shown in the figure, extend  $BC$  and  $EF$  to draw a rectangle  $PQRS$  enclosing the hexagon. Since opposite sides of the hexagon are parallel, opposite angles are equal (i.e.  $\angle A = \angle D$ ,  $\angle B = \angle E$  and  $\angle C = \angle F$ ). Let  $a, b, c, d, e, f$  denote the lengths of  $AB, BC, CD, DE, EF$  and  $FA$  respectively. We have



$$\begin{aligned} 2BF &\geq PA + AS + QD + DR \\ &= a \sin B + f \sin F + c \sin C + d \sin E \\ &= a \sin B + f \sin C + c \sin C + d \sin B \end{aligned}$$

Hence

$$R_A = \frac{BF}{2 \sin A} = \frac{1}{4} \left( \frac{a \sin B}{\sin A} + \frac{f \sin C}{\sin A} + \frac{c \sin C}{\sin A} + \frac{d \sin B}{\sin A} \right).$$

Similarly, we have

$$R_C = \frac{1}{4} \left( \frac{c \sin A}{\sin C} + \frac{b \sin B}{\sin C} + \frac{e \sin B}{\sin C} + \frac{f \sin A}{\sin C} \right)$$

and

$$R_E = \frac{1}{4} \left( \frac{e \sin C}{\sin B} + \frac{d \sin A}{\sin B} + \frac{a \sin A}{\sin B} + \frac{b \sin C}{\sin B} \right).$$

Adding these inequalities and using the fact that  $\frac{x}{y} + \frac{y}{x} \geq 2 \sqrt{\frac{x}{y} \cdot \frac{y}{x}} = 2$  for  $x, y > 0$ , we have

$$R_A + R_C + R_E \geq \frac{1}{4} (2a + 2b + 2c + 2d + 2e + 2f) = \frac{p}{2}$$

as desired.