Hausdorff Dimension

Hausdorff dimension was singled out by Mandelbrot when he defined “fractal” as a set for which the Hausdorff dimension is strictly greater than the topological dimension. Before going through the details, let us agree with the following:

Notation and Terminology.

The *diameter* of a non-empty set U (in $\mathbb{R}^n$, or any metric space) is

$$|U| \triangleq \sup\{|x-y| : x, y \in U\}.$$

A countable collection $\{U_i\}$ of subsets of $\mathbb{R}^n$ is said to be a d-cover of $F \subseteq \mathbb{R}^n$ if

$$F \subseteq \bigcup_i U_i \qquad \text{and} \qquad 0 < |U_i| < d \quad \forall i.$$

Definition. (Hausdorff measure)

Given $s \geq 0$. For each $F \subseteq \mathbb{R}^n$, denote

$$\mathsf{H}_d^s(F) = \inf\left\{\sum_i |U_i|^s : \{U_i\} \text{is a } d\text{-cover of } F\right\}.$$

Note that $\mathsf{H}_d^s(F)$ increases as d decreases, this allows us to define

$$\mathsf{H}^s(F) = \lim_{d\to 0} \mathsf{H}_d^s(F).$$

We call $\mathsf{H}^s : P(\mathbb{R}^n) \to [0,\infty]$ the s-dimensional Hausdorff measure.

In order to justify the term “measure”, we have the following

Theorem 1.

The set function $\mathsf{H}^s : P(\mathbb{R}^n) \to [0,\infty]$ is an outer measure on $\mathbb{R}^n$ and the Borel sets are measurable, i.e.

(1) $\mathsf{H}^s(f) = 0$.

(2) $\mathsf{H}^s(E) \leq \mathsf{H}^s(F)$ if $E \subseteq F$.

(3) If $F_1, F_2, \ldots$ is a sequence of subsets of $\mathbb{R}^n$, then

$$\mathsf{H}^s\left(\bigcup_{k=1}^{\infty} F_k\right) \leq \sum_{k=1}^{\infty} \mathsf{H}^s(F_k).$$

(4) If F is Borel, then for any $A \subseteq \mathbb{R}^n$,

$$\mathsf{H}^s(A) = \mathsf{H}^s(A \cap F) + \mathsf{H}^s(A \cap F^c).$$

- (2) is called *monotonicity* of (outer) measure, (3) is *countable sub-additivity.* (4) guarantees the equality in (3) holds if $F_1, F_2, \ldots$ are mutually disjoint Borel sets.

Proof.

Checking (1)

This is obvious because any collection of sets is a cover of empty set.

Checking (2)

Simply note that any d-cover of F is also d-cover of E, and the infimum over a larger set is smaller.

Checking (3)

Assume $\mathrm{H}^s(F_k) < \infty$ for all k. Given $e > 0$, for any $k \in \mathbb{N}$, there exists a d-cover $\{U_i^{(k)}\}$ (depending on k) of F_k such that

$$\sum_i \left|U_i^{(k)}\right|^s < \mathrm{H}_d^s(F_k) + \frac{e}{2^k}.$$

Taking summation through $k = 1, 2, \ldots,$

$$\sum_k \sum_i \left|U_i^{(k)}\right|^s < \sum_{k=1}^{\infty} \mathrm{H}_d^s(F_k) + e.$$

Since $\{U_i^{(k)}\}$ is a cover of F_k for each k, the collection $\bigcup_{k\in\mathbb{N}} \{U_i^{(k)}\}$ is a cover of $\bigcup_{k\in\mathbb{N}} F_k$, it follows that

$$\mathrm{H}_d^s\left(\bigcup_{k\in\mathbb{N}} F_k\right) \le \sum_k \sum_i \left|U_i^{(k)}\right|^s < \sum_{k=1}^{\infty} \mathrm{H}_d^s(F_k) + e.$$

Letting $e \to 0$,

$$\mathrm{H}_d^s\left(\bigcup_{k\in\mathbb{N}} F_k\right) \le \sum_{k=1}^{\infty} \mathrm{H}_d^s(F_k)$$

for any $d > 0$. We let $d \to 0$ and get

$$\mathrm{H}^s\left(\bigcup_{k\in\mathbb{N}} F_k\right) \le \sum_{k=1}^{\infty} \mathrm{H}^s(F_k).$$

Now, H^s is shown to be an outer measure. Our goal is to prove the Borel sets are measurable. Before proving (4), we need a lemma.

Lemma. If $X, Y \subseteq \mathbb{R}^n$ and $d(X,Y) > 0$, then $\mathrm{H}^s(X \cup Y) = \mathrm{H}^s(X) + \mathrm{H}^s(Y)$.

Reason. For sufficiently small $d > 0$, no set with diameter less than d can intersect both X and Y. The lemma follows from this observation.

Checking (4)

Since the measurable sets form a s–algebra, we assume (without loss of generality) that $F = \overline{B}(x,r)$ is a closed ball centered at x with radius r.

By (3), H^s is known to be countable sub-additive. It remains to show

$$\mathrm{H}^s(A) \ge \mathrm{H}^s(A \cap F) + \mathrm{H}^s(A \cap F^c) \qquad \text{----------------------} \qquad (1)$$

for any $A \subseteq \mathbb{R}^n$.

Let $A_k = \left\{ y \in A : r + \dfrac{1}{k+1} \le |x - y| < r + \dfrac{1}{k} \right\}$ for $k = 0, 1, 2\ldots$, then

$$A \cap F^c = \bigcup_{k=0}^{\infty} A_k.$$

Here we have used the condition that F is closed.

Now, for each $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{H}^s(A) &= \mathcal{H}^s\left((A \cap F) \cup \bigcup_{k=0}^{\infty} A_k \right) \\ &\geq \mathcal{H}^s\left((A \cap F) \cup \bigcup_{k=0}^{n} A_k \right) \\ &= \mathcal{H}^s(A \cap F) + \mathcal{H}^s\left(\bigcup_{k=0}^{n} A_k \right) \end{aligned}$$

the last equality holds because of the lemma. Letting $n \to \infty$,

$$\mathcal{H}^s(A) \geq \mathcal{H}^s(A \cap F) + \lim_{n\to\infty} \mathcal{H}^s\left(\bigcup_{k=0}^{n} A_k \right) \qquad \text{-------------------} \qquad (2)$$

We claim that $\lim_{n\to\infty} \mathcal{H}^s\left(\bigcup_{k=0}^{n} A_k \right) = \mathcal{H}^s\left(\bigcup_{k=0}^{\infty} A_k \right)$. If this is done, then (2) is equivalent to (1) and our proof is completed.

The inequality $\lim_{n\to\infty} \mathcal{H}^s\left(\bigcup_{k=0}^{n} A_k \right) \leq \mathcal{H}^s\left(\bigcup_{k=0}^{\infty} A_k \right)$ is trivial (because $\mathcal{H}^s$ has monotonicity). To get the opposite result, note that for each $n \in \mathbb{N}$ one has

$$\begin{aligned} \mathcal{H}^s\left(\bigcup_{k=0}^{\infty} A_k \right) &\leq \mathcal{H}^s\left(\bigcup_{k=0}^{n} A_k \right) + \mathcal{H}^s\left(\bigcup_{k=1}^{\infty} A_{n+2k-1} \right) + \mathcal{H}^s\left(\bigcup_{k=1}^{\infty} A_{n+2k} \right) \\ &= \mathcal{H}^s\left(\bigcup_{k=0}^{n} A_k \right) + \sum_{k=1}^{\infty} \mathcal{H}^s(A_{n+2k-1}) + \sum_{k=1}^{\infty} \mathcal{H}^s(A_{n+2k}) \end{aligned}$$

Be careful that our lemma is used again to guarantee $\mathcal{H}^s\left(\bigcup_{k=1}^{\infty} A_{n+2k-1} \right) = \sum_{k=1}^{\infty} \mathcal{H}^s(A_{n+2k-1})$ and $\mathcal{H}^s\left(\bigcup_{k=1}^{\infty} A_{n+2k} \right) = \sum_{k=1}^{\infty} \mathcal{H}^s(A_{n+2k})$.

Now, letting $n \to \infty$ in the previous inequality,

$$\mathcal{H}^s\left(\bigcup_{k=0}^{\infty} A_k \right) \leq \lim_{n\to\infty} \mathcal{H}^s\left(\bigcup_{k=0}^{n} A_k \right).$$

This proves our claim and completes the proof.

Q.E.D.

Sometimes we use the open cover in the definition of Hausdorff measure. More precisely, if we define

$$\mathcal{M}_d^s(F) = \inf \left\{ \sum_i |U_i|^s : \{U_i\} \text{is a } d\text{-cover of } F \text{ by open sets} \right\}$$

and

$$\mathcal{M}^s(F) = \lim_{d\to 0} \mathcal{M}_d^s(F),$$

then $\mathsf{M}^s(F) = \mathsf{H}^s(F)$ for all $F \subseteq \mathbb{R}^n$. The proof goes as follows:

Clearly, $\mathsf{M}^s(F) \geq \mathsf{H}^s(F)$ since any open d-cover of F is a permissible covering in the definition of H^s_d. To show the opposite inequality, we suppose $\mathsf{H}^s(F) < \infty$ and choose an $e > 0$. Then for each $d > 0$ there is a d-cover $\{U_i\}$ of F such that

$$\sum_i |U_i|^s < \mathsf{H}^s_d(F) + e \qquad \text{------------------------------} \tag{3}$$

For each i, let $V_i = \bigcup_{x \in U_i} B(x,r)$, where $r > 0$ is chosen to be sufficiently small so that

$$|V_i| < d \qquad \text{and} \qquad |V_i|^s \leq |U_i|^s + \frac{e}{2^i}.$$

Then

$$\sum_i |V_i|^s \leq \sum_i |U_i|^s + e \qquad \text{-------------------------------} \tag{4}$$

By (3) and (4), $\sum_i |V_i|^s < \mathsf{H}^s_d(F) + 2e$. Note that $\{V_i\}$ is a d-cover of F by open sets, it follows that

$$\mathsf{M}^s_d(F) \leq \sum_i |V_i|^s < \mathsf{H}^s_d(F) + 2e$$

true for arbitrary $e > 0$. So, $\mathsf{M}^s_d(F) \leq \mathsf{H}^s_d(F)$ for all $d > 0$, follows that $\mathsf{M}^s(F) \leq \mathsf{H}^s(F)$. Hence, $\mathsf{M}^s(F) = \mathsf{H}^s(F)$ for all $F \subseteq \mathbb{R}^n$.

- It is easy to check that the same value for Hausdorff measure will give if we use only d-cover by closed sets in the definition. This is because the closure of a set U has the same diameter as U, i.e. $|\overline{U}| = |U|$.

- If F is compact, we may even consider only finite d-cover of F in the definition of Hausdorff measure. This gives the same value because any d-cover can be expanded slightly to open d-cover and then reduce to finite subcover.

It is not difficult to see H^0 is the counting measure. More generally, we have

Theorem 2.
If F is a Borel subset of $\mathbb{R}^n$, then

$$\mathsf{H}^n(F) = c_n \mathrm{vol}^n(F),$$

where the constant $c_n = \dfrac{p^{n/2}}{2^n (n/2)!}$.

For example, $\mathsf{H}^1(F) = \text{length}(F)$, $\mathsf{H}^2(F) = \dfrac{p}{4} \times \text{area}(F)$, $\mathsf{H}^3(F) = \dfrac{4p}{3} \times \text{vol}(F)$.

The proof of this theorem is not important for our purpose and hence omitted.

Theorem 3. **(Scaling Property)**
For any $F \subseteq \mathbb{R}^n$ and $l > 0$, one has
$$H^s(lF) = l^s H^s(F).$$

Proof.
Suppose $\{U_i\}$ is a d-cover of F. Then $\{lU_i\}$ is a ld-cover of lF. Hence,
$$H^s_{ld}(lF) \le \sum_i |lU_i|^s = l^s \sum_i |U_i|^s .$$
Take infimum, run through all d-covers $\{U_i\}$ of F, we get
$$H^s_{ld}(lF) \le l^s H^s_d(F).$$
Letting $d \to 0$,
$$H^s(lF) \le l^s H^s(F).$$
Replace l by $\dfrac{1}{l}$ and F by lF, the opposite inequality follows.

Q.E.D.

We expect H^s to be translation invariant, i.e. $H^s(F+x) = H^s(F)$. The following theorem explains this.

Theorem 4.
Let $F \subseteq \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ be a mapping such that
$$|f(x) - f(y)| \le c|x-y|^a \qquad (x, y \in F)$$
for positive constants c, a. Then for each s,
$$H^{s/a}(f(F)) \le c^{s/a} H^s(F).$$

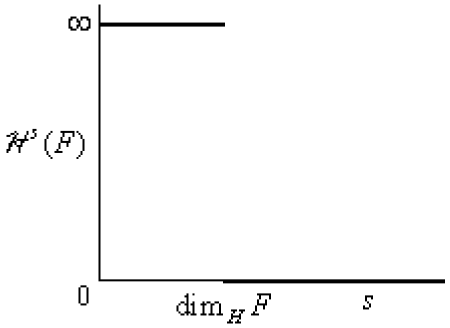
Proof.
Suppose $\{U_i\}$ is a d-cover of F, then $|f(F \cap U_i)| \le c|U_i|^a < cd^a$. That means $\{f(F \cap U_i)\}$ is a cd^a-cover of $f(F)$. So,
$$H^{s/a}_{cd^a}(f(F)) \le \sum_i |f(F \cap U_i)|^{s/a} \le \sum_i \left(c|U_i|^a\right)^{s/a} = c^{s/a} \sum_i |U_i|^s .$$
Taking infimum,
$$H^{s/a}_{cd^a}(f(F)) \le c^{s/a} H^s_d(F).$$
The result follows by letting $d \to 0$.

Q.E.D.

- A function satisfying
$$|f(x) - f(y)| \le c|x-y| \qquad (x, y \in F)$$
is called *Lipschitz function*. In this case we have $H^s(f(F)) \le c^s H^s(F)$.

- If f is an isometry, i.e. $|f(x) - f(y)| = |x-y|$, then $H^s(f(F)) = H^s(F)$.

For a fixed $F \in \mathbb{R}^n$, we can calculate $\mathcal{H}^s(F)$ for various values of s and plot the following graph.



The graph of $\mathcal{H}^s(F)$ is decreasing because $\mathcal{H}^s_d(F)$ is decreasing with s for $d < 1$. Note also that the value of $\mathcal{H}^s(F)$ "jump" from ∞ to 0 at a particular point. The reason is simple: if $t > s$, then for any d-cover $\{U_i\}$ of F we have

$$\sum_i |U_i|^t \le d^{t-s} \sum_i |U_i|^s .$$

Taking infimum, we get $\mathcal{H}^t_d(F) \le d^{t-s} \mathcal{H}^s_d(F)$. If $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$. This explains why the graph "jump" from ∞ to 0.

Definition. (Hausdorff Dimension)

Let $F \in \mathbb{R}^n$, the Hausdorff dimension (or Hausdorff-Besicovitch dimension) is

$$\dim_H F \triangleq \inf\left\{s : \mathcal{H}^s(F) = 0\right\} = \sup\left\{s : \mathcal{H}^s(F) = \infty\right\}.$$

- From the above discussion, we see that

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{if } s < \dim_H F \\ 0 & \text{if } s > \dim_H F \end{cases}$$

At the point $s = \dim_H F$, the value of $\mathcal{H}^s(F)$ may be zero, infinity, or satisfies

$$0 < \mathcal{H}^s(F) < \infty .$$

Theorem 5.

Let $F \in \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ satisfying

$$|f(x) - f(y)| \le c|x - y|^a \qquad (x, y \in F)$$

for positive constants c, a. Then $\dim_H f(F) \le (1/a)\dim_H F$.

Proof. Make use of Theorem 4.

- If f is Lipschitz function (i.e. $a = 1$), then $\dim_H f(F) \le \dim_H F$.

- If f is bi-Lipschitz, i.e. $c_1|x - y| \le |f(x) - f(y)| \le c_2|x - y| \quad (x, y \in F)$ for $0 < c_1 \le c_2 < \infty$, then $\dim_H f(F) = \dim_H F$.

Calculation of Hausdorff Dimension

Example 1. (Cantor Set)
Let C be the middle third Cantor set.

$C_0 = [0,1]$

$C_1 = [0,1/3] \cup [2/3,1]$

$C_2 = [0,1/9] \cup \cdots \cup [8/9,1]$

Claim: $\dim_H C = \dfrac{\log 2}{\log 3}$. At the critical point $s = \dim_H S$, we have $\dfrac{1}{2} \leq \mathcal{H}^s(C) \leq 1$.

Proof.

Step 1 Upper bound of $\mathcal{H}^s(C)$

There is an obvious covering $\{U_i\}$ of C using 2^k "basic intervals" of length 3^{-k}. For this 3^{-k}-cover of C, one has

$$\mathcal{H}^s_{3^{-k}}(C) \leq \sum_i |U_i|^s = 2^k \times (3^{-k})^s = 1 \quad (\because\ 2 = 3^s \text{ by our choice of } s)$$

Letting $k \to \infty$, $\mathcal{H}^s(C) \leq 1$.

Step 2 Lower bound of $\mathcal{H}^s(C)$

We will show

$$\sum_i |U_i|^s \geq \frac{1}{2} \quad \text{----------------------------------} \quad (5)$$

for any cover $\{U_i\}$ of C. Since C is compact, it suffices to verify (5) for finite collection of closed sets $\{U_i\}$ covering C.

Let $\{U_i\}$ be such a finite closed cover. For each U_i, let $k \in \mathbb{Z}$ so that

$$3^{-(k+1)} \leq |U_i| < 3^{-k}$$

Note that U_i intersects with at most 1 basic interval of C_k. If $j \geq k$, then the number of basic intervals of C_j that intersect U_i is at most

$$\begin{aligned} 2^{j-k} &= 2^j \times 3^{-sk} \\ &= 2^j \times 3^s \times 3^{-s(k+1)} \\ &\leq 2^j \times 3^s \times |U_i|^s \end{aligned}$$

The above inequality holds for a particular U_i (recall that k depends on U_i). Since there are only finitely many U_i's, we can choose j to be sufficiently large so that the inequality true for all U_i's. Since $\{U_i\}$ intersects all 2^j basic intervals of C_j, counting intervals gives

$$2^j \leq 2^j \times 3^s \times \sum_i |U_i|^s .$$

Equivalently,

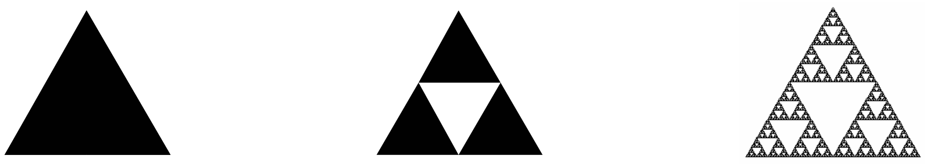
$$\sum_i |U_i|^s \geq 3^{-s} = \frac{1}{2}.$$

This proves (5).

Q.E.D.

Example 2. (Sierpinski Triangle)

The Sierpinski Triangle *S* is defined to be the intersection $\bigcap_{k=0}^{\infty} S_k$, where S_0 is a closed set containing all points inside and on the boundary of an equilateral triangle, and S_k $(k \geq 1)$ is obtained by removing all points inside the medial triangles of each "basic triangle" of S_{k-1} (see the figure below).



S_0 S_1 S

Claim: $\dim_H S = \dfrac{\log 3}{\log 2}$. At the critical point $s = \dim_H S$, we have $\dfrac{1}{18} \leq \mathcal{H}^s(S) \leq 1$.

Proof.

Without loss of generality, we assume the largest triangle has side 1.

<u>Step 1</u> Upper bound of $\mathcal{H}^s(S)$

Taking the obvious covering $\{U_i\}$ of *S* by 3^k basic triangles of side 2^{-k} (note that the diameter is also 2^{-k}). For this 2^{-k}-cover of *S*, one has

$$\mathcal{H}^s_{2^{-k}}(S) \leq \sum_i |U_i|^s = 3^k \times (2^{-k})^s = 1 \quad (\because 3 = 2^s \text{ by our choice of } s)$$

Letting $k \to \infty$, $\mathcal{H}^s(S) \leq 1$.

<u>Step 2</u> Lower bound of $\mathcal{H}^s(S)$

We will show

$$\sum_i |U_i|^s \geq \frac{1}{18} \quad \text{----------------------------------} \quad (6)$$

for any cover $\{U_i\}$ of *S*. Since *S* is compact, it suffices to verify (6) for finite collection of closed sets $\{U_i\}$ covering *S*.

Let $\{U_i\}$ be such a finite closed cover. For each U_i, let $k \in \mathbb{Z}$ so that

$$2^{-(k+2)} \leq |U_i| < 2^{-(k+1)}$$

Then U_i intersects with at most 2 basic triangles of S_k (why?). If $j \geq k$, the number of basic triangles of S_j that intersect U_i is at most

$$\begin{aligned} 2\times 3^{j-k} &= 3^j \times 2^{-sk+1} \\ &= 3^j \times 2^{2s+1} \times 2^{-s(k+2)} \\ &\le 3^j \times 18 \times |U_i|^s \end{aligned}$$

The above inequality holds for a particular U_i (recall that k depends on U_i). Since there are only finitely many U_i's, we can choose j to be sufficiently large so that the inequality true for all U_i's. Since $\{U_i\}$ intersects all 3^j basic triangles of S_j, counting triangles gives

$$3^j \le 3^j \times 18 \times \sum_i |U_i|^s .$$

Hence, $\sum_i |U_i|^s \ge \frac{1}{18}$ and completes the proof.

Q.E.D.

- If we assume that $0 < \mathcal{H}^s(S) < \infty$ at the critical point $s = \dim_H S$ (a big assumption, although this happens very often), there is a simple method to find $\dim_H S$: note that S can split into

  $$S = S_1 \cup S_2 \cup S_3,$$

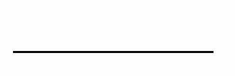
  where S_1, S_2, S_3 are geometrically similar to S. Then

  $$\begin{aligned} \mathcal{H}^s(S) &= \mathcal{H}^s(S_1) + \mathcal{H}^s(S_2) + \mathcal{H}^s(S_3) \\ &= \frac{1}{2^s}\mathcal{H}^s(S) + \frac{1}{2^s}\mathcal{H}^s(S) + \frac{1}{2^s}\mathcal{H}^s(S) \\ &= \frac{3}{2^s}\mathcal{H}^s(S) \end{aligned}$$

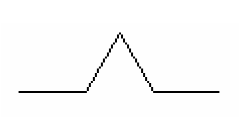
  Since $0 < \mathcal{H}^s(S) < \infty$, we have $\frac{3}{2^s} = 1$ and hence $s = \frac{\log 3}{\log 2}$.

Example 3. (Koch Curve)

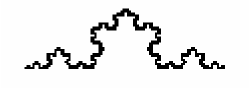
Roughly speaking, the Koch Curve is defined as follows (precise definition omitted):



$K_0 = [0,1]$



K_1



K

Claim: $\dim_H K = \frac{\log 4}{\log 3}$. At the critical point $s = \dim_H S$, we have $2^{-(s+2)} \le \mathcal{H}^s(S) \le 1$.

Proof.

As before, the upper bound of $\mathcal{H}^s(K)$ is easy. For the lower bound, we need to show

$$\sum_i |U_i|^s \ge 2^{-(s+2)}$$

for any cover $\{U_i\}$ of K. Since K is compact, it suffices consider finite collection of closed sets $\{U_i\}$ covering K.

Let $\{U_i\}$ be such a finite closed cover. For each U_i, let $k \in \mathbb{Z}$ so that

$$3^{-(k+1)} \le \frac{2}{\sqrt{3}} \times |U_i| < 3^{-k}$$

Then U_i intersects with at most 2 basic line segments of K_k (why?). If $j \ge k$, the number of basic line segments of K_j that intersect U_i is at most

$$\begin{aligned} 2 \times 4^{j-k} &= 4^j \times 2 \times 3^{-sk} \\ &= 4^j \times 2 \times 3^s \times 3^{-s(k+1)} \\ &\le 4^j \times 2 \times 3^s \times \left(\frac{2}{\sqrt{3}} \times |U_i| \right)^s \\ &= 4^j \times 2^{s+1} \times 3^{s/2} \times |U_i|^s \\ &= 4^j \times 2^{s+2} \times |U_i|^s \end{aligned}$$

Again, choose j to be sufficiently large and count the line segments, one has

$$4^j \le 4^j \times 2^{s+2} \times \sum_i |U_i|^s .$$

Equivalently,

$$\sum_i |U_i|^s \ge 2^{-(s+2)} .$$

Q.E.D.

References

[1] Kenneth Falconer, *Fractal Geometry — Mathematical Foundations and Applications*, John Wiley & Sons.

[2] Gerald A. Edgar, *Measure, Topology, and Fractal Geometry*, Springer-Verlag.