Integration on $\mathbb{R}^n$

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January 4, 2003

Definition and Basic Properties

Let $A \subseteq \mathbb{R}^n$ be a closed rectangle of the form $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Recall that a partition of the interval $[a_i, b_i]$ means a finite set

$$P_i = \{\, a_i = x_0 < x_1 < \cdots < x_{k-1} < x_k = b_i \,\}.$$

We now define the term **partition of A** to be a set of the form

$$P = P_1 \times \cdots \times P_n,$$

where each P_i is an ordinary partition of $[a_i, b_i]$. This partition P divides A into finitely many closed subrectangles S in the natural way.

Notation. Suppose a bounded function $f\colon A \to \mathbb{R}$ and a partition P of A are given. For each subrectangle S, denote

$$m_S(f) := \inf_{x\in S} f(x) \qquad \text{and} \qquad M_S(f) := \sup_{x\in S} f(x).$$

These notations will be used throughout this article whenever the partition is clear from the context and there is no risk of confusion. Having $m_S(f)$ and $M_S(f)$ in hand, we define further that

$$L(f,P) := \sum_S m_S(f)\operatorname{vol}^n(S), \tag{1}$$

$$U(f,P) := \sum_S M_S(f)\operatorname{vol}^n(S). \tag{2}$$

where the summations run through all subrectangles S. The volume of S, denoted by $\operatorname{vol}^n(S)$ as in (1) and (2), is defined by the trivial way.

$L(f,P)$ and $U(f,P)$ are called the **lower sum** and **upper sum** of f for the partition P. It is obvious from (1) and (2) that $L(f,P) \le U(f,P)$.

Notation. We say that a partition P' refines P (or P' is a refinement of P) if every subrectange of P' is contained in a subrectangle of P. For any two partitions P_1, P_2, there exists **the smallest common refinement** of P_1, P_2 and denoted by $P_1 \bigstar P_2$.

Lemma 1. *If P' refines P, then $L(f,P) \leq L(f,P') \leq U(f,P') \leq U(f,P)$.*

Lemma 2. *For any two partitions P_1, P_2, one has $L(f,P_1) \leq U(f,P_2)$.*

The proofs of Lemma 1 and 2 are not difficult and are left to reader as exercise. Now, we are in the position to define the integral $\int_A f$.

Definition. Let $A \subseteq \mathbb{R}^n$ be a closed rectangle and $f\colon A \to \mathbb{R}$ be bounded. The **lower/upper integral** of f are defined to be

$$\underline{\int_A} f := \sup L(f,P) \qquad \text{and} \qquad \overline{\int_A} f := \inf U(f,P)$$

respectively, where the supremum/infimum is taken over all possible partitions of A.

By Lemma 2, one has $\underline{\int}_A f \leq \overline{\int}_A f$. If it happens that $\underline{\int}_A f = \overline{\int}_A f$, the function f is said to be **integrable on** A. The common value is called the **integral of** f **over** A, denoted by $\int_A f(x^1, \ldots, x^n)\, dx^1 \cdots dx^n$ or simply by $\int_A f$. If $f\colon [a,b] \to \mathbb{R}$, then $\int_{[a,b]} f = \int_a^b f$ is our usual notation in one variable calculus.

Notation. Let us denote the set of all integrable functions on A by the symbol $\mathfrak{R}(A)$.

Here is a very useful criterion for integrability.

Proposition 3. *A bounded function $f\colon A \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$ there exists a partition P such that*

$$U(f,P) - L(f,P) < \epsilon. \tag{3}$$

Proof. Suppose for any $\epsilon > 0$ there is a partition P satisfying (3). Then

$$\overline{\int_A} f - \underline{\int_A} f \leq U(f,P) - L(f,P) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\overline{\int}_A f = \underline{\int}_A f$ and $f \in \mathfrak{R}(A)$.

On the other hand, if $f \in \mathfrak{R}(A)$ and $\epsilon > 0$ is given, then there exist partitions P_1, P_2 such that

$$\int_A f - \frac{\epsilon}{2} < L(f,P_1) \qquad \text{and} \qquad U(f,P_2) < \int_A f + \frac{\epsilon}{2}.$$

Take $P = P_1 \bigstar P_2$, then

$$U(f,P) - L(f,P) \leq U(f,P_2) - L(f,P_1) < \left(\int_A f + \frac{\epsilon}{2}\right) - \left(\int_A f - \frac{\epsilon}{2}\right) = \epsilon.$$

□

One of the most important consequences of Theorem 3 is the following.

Proposition 4. *Continuous function $f\colon A \to \mathbb{R}$ is integrable.*

Proof. Since f is continuous, it is bounded. Let $\epsilon > 0$ be given. Since A is compact, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \qquad \text{whenever } x,y \in A \text{ and } |x-y| < \delta.$$

Choose a partition P such that each subrectangle of P has diameter less than δ. Then $M_S(f) - m_S(f) \leq \epsilon$ for each subrectangle S, so

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_S (M_S(f) - m_S(f))\text{vol}^n(S) \\ &\leq \epsilon \cdot \text{vol}^n(A). \end{aligned}$$

As $\epsilon > 0$ is arbitrary, $f \in \mathfrak{R}(A)$ by Theorem 3. □

Theorem 5 (Monotonicity). *If $f, g \in \mathfrak{R}(A)$ and $f \leq g$, then $\int_A f \leq \int_A g$.*

Proof. Simply note that $L(f,P) \leq L(g,P)$ for any partition P. □

Theorem 6 (Linearity). *If $f, g \in \mathfrak{R}(A)$ and $c \in \mathbb{R}$, then $f+g, cf \in \mathfrak{R}(A)$. Moreover,*

$$\int_A (f+g) = \int_A f + \int_A g \qquad \text{and} \qquad \int_A (cf) = c\int_A f.$$

Proof. For any partition P and any subrectangle S, one has $m_S(f+g) \geq m_S(f) + m_S(g)$. So, $L(f+g,P) \geq L(f,P) + L(g,P)$ and follows that

(4) $$\underline{\int}_A (f+g) \geq L(f,P) + L(g,P) \qquad \text{for all partition P.}$$

For any two partitions P_1, P_2, apply (4) to $P = P_1 \bigstar P_2$ and get

(5) $$\underline{\int}_A (f+g) \geq L(f,P) + L(g,P) \geq L(f,P_1) + L(g,P_2).$$

Since P_1, P_2 are arbitrary, taking supremum in (5) over all possible P_1, P_2 leads to

(6) $$\underline{\int}_A (f+g) \geq \int_A f + \int_A g.$$

Similar argument shows

(7) $$\overline{\int}_A (f+g) \leq \int_A f + \int_A g.$$

Now, (6) and (7) give

$$\underline{\int}_A f = \overline{\int}_A f = \int_A f + \int_A g.$$

It remains to prove $\int_A (cf) = c\int_A f$. The case $c \geq 0$ is easy because $m_S(cf) = cm_S(F)$. The case $c = -1$ is also straightforward since $m_S(-f) = -M_S(f)$. The general cases follow by these. □

Exercise. Let $f\colon A \to \mathbb{R}$ and let P be any partition of A. Prove that $f \in \mathfrak{R}(A)$ if and only if $f \in \mathfrak{R}(S)$ for each subrectangle S. In that case, $\int_A f = \sum_S \int_S f$.

Lebesgue Criterion for Integrability

We begin this section by the following definition.

Definition (Outer Measure). Given a set $F \subseteq \mathbb{R}^n$ (not necessary a rectangle). The **outer measure** of F is defined by

$$m^*(F) := \inf \left\{ \sum_{i=1}^{\infty} \operatorname{vol}^n(U_i) \,\middle|\, \text{each } U_i \text{ is open rectangle and } F \subseteq \bigcup_{i=1}^{\infty} U_i \right\}.$$

Here are the basic properties of m^*.

(i) $m^*(F) = 0$ if F is countable;

(ii) $m^*(F) = \operatorname{vol}^n(F)$ if F is rectangle (open or closed);

(iii) $m^*(E) \leq m^*(F)$ whenever $E \subseteq F$;

(iv) $m^*\left(\bigcup_{i=1}^{\infty} F_i\right) \leq \sum_{i=1}^{\infty} m^*(F_i)$ where $F_1, F_2, \cdots \subseteq \mathbb{R}^n$;

(v) There exists a σ-algebra $\mathfrak{M}$ containing all Borel sets such that

$$m^*\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} m^*(F_i)$$

holds whenever $F_1, F_2, \ldots$ are disjoint members of $\mathfrak{M}$.

Remark. (iii) is called **monotonicity** of m^* and (iv) is the **countable subadditivity**. An important corollary of (iv) is that countable union of measure zero sets has measure zero.

The set function $m^*\colon \mathcal{P}(\mathbb{R}) \to [0, \infty]$ is called **Lebesgue outer measure**[1]. It can be used to define a more general kind of integral, namely, Lebesgue integral. This leads to another story which we are not going to discuss here.

We defined m^* in this section as a tool to introduce an elegant theorem concerning the sufficient and necessary condition for a function to be integrable. Before stating this theorem, let's see a lemma first.

Lemma 7. *Let $A \subseteq \mathbb{R}^n$ be a closed rectangle and $f\colon A \to \mathbb{R}$ be bounded. If $o(f, x) < \epsilon$ for all $x \in A$, then there is a partition P of A such that $U(f, P) - L(f, P) < \epsilon \cdot \mathrm{vol}^n(A)$.*

Remark. The oscillation $o(f, x)$ is defined to be

$$o(f, x) := \lim_{\delta \to 0} \left(\sup_{\substack{y \in A \\ |x-y|<\delta}} f(y) - \inf_{\substack{y \in A \\ |x-y|<\delta}} f(y) \right).$$

Proof. For each $x \in A$, there is a closed rectangle S_x containing x as its interior such that $M_{S_x}(f) - m_{S_x}(f) < \epsilon$. Since A is compact, there exists finitely many $S_{x_1}, S_{x_2}, \ldots, S_{x_k}$ that cover A. Take a partition P so that each subrectangle of P is contained in some S_{x_i}, the result follows. □

The next theorem is one of the main results in this article.

Theorem 8 (Lebesgue Criterion). *Let $A \subseteq \mathbb{R}^n$ be a closed rectangle and $f\colon A \to \mathbb{R}$ be bounded. Let $B = \{\, x \in A \mid f \text{ is discontinuous at } x \,\}$. We have $f \in \mathfrak{R}(A)$ if and only if $m^*(B) = 0$.*

Proof. Note that $B = \{\, x \in A \mid o(f, x) > 0 \,\}$. For each $j \in \mathbb{N}$, write

$$B_j = \left\{ x \in A \;\middle|\; o(f, x) \geq \frac{1}{j} \right\}.$$

[1] It's restriction to $\mathfrak{M}$ is called Lebesgue measure and denoted by m.

Then each B_j is compact (why?) and $B = B_1 \cup B_2 \cup \cdots$. By the monotonicity and countable subadditivity of m^*, $m^*(B) = 0$ if and only if $m^*(B_j) = 0$ for each j.

Suppose now $f \in \mathfrak{R}(A)$ and $\epsilon > 0$ is given. Choose a partition P so that $U(f,P) - L(f,P) < \epsilon$. Let $\mathfrak{S}_j$ be the collection of subrectangles S that it's interior intersects B_j. If $S \in \mathfrak{S}_j$, then $M_S(f) - m_S(f) \geq \frac{1}{j}$ and follows that

$$\begin{aligned} m^*(B_j) &= m^*\left(B_j \cap \bigcup_S S^\circ\right) \\ &\leq \sum_{S \in \mathfrak{S}_j} \mathrm{vol}^n(S) \\ &\leq j \sum_{S \in \mathfrak{S}_j} (M_S(f) - m_S(f))\mathrm{vol}^n(S) \\ &\leq j\,(U(f,P) - L(f,P)) \\ &< j\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $m^*(B_j) = 0$. This is true for all j, so $m^*(B) = 0$.

Conversely, suppose $m^*(B_j) = 0$ for each j. If $\epsilon > 0$, choose $j \in \mathbb{N}$ be such that $\frac{1}{j} < \epsilon$. For this j, there is a countable open rectangle cover $\{\, U_i \,\}$ of B_j with $\sum_i \mathrm{vol}^n(U_i) < \epsilon$. Since B_j is compact, we may assume $\{\, U_i \,\} = \{\, U_1, U_2, \ldots, U_k \,\}$ is a finite cover. Now, choose a partition P so that its subrectangles can be divided into two classes:

(i) $\mathfrak{S}$, consists of those S such that $S \subseteq \overline{U_i}$ for some i;

(ii) $\mathfrak{S}'$, consists of those S with $S \cap B_j = \emptyset$.

Note that, if $S \in \mathfrak{S}'$, then $o(f,x) < \frac{1}{j} < \epsilon$ for all $x \in S$. Lemma 7 allows us to assume further that

$$\sum_{S \in \mathfrak{S}'} (M_S(f) - m_S(f))\,\mathrm{vol}^n(S) < \epsilon \cdot \mathrm{vol}^n(A). \tag{8}$$

Let $M > 0$ be constant such that $|f(x)| \leq M$ for all $x \in A$. Then

$$\sum_{S \in \mathfrak{S}} (M_S(f) - m_S(f))\,\mathrm{vol}^n(S) \leq 2M \sum_{i=1}^{k} \mathrm{vol}^n(U_i) < 2M\epsilon. \tag{9}$$

Finally, (8) and (9) together give

$$U(f,P) - L(f,P) < \epsilon(2M + \mathrm{vol}^n(A)).$$

Since $\epsilon > 0$ is arbitrary, we conclude that $f \in \mathfrak{R}(A)$. □

Corollary 8.1. *If $f, g \in \mathfrak{R}(A)$, then $fg \in \mathfrak{R}(A)$.*

Corollary 8.2. *If $f\colon A \to \mathbb{R}$ is in $\mathfrak{R}(A)$ and $g\colon f(A) \to \mathbb{R}$ is continuous, then $g \circ f \in \mathfrak{R}(A)$. In particular, $|f| \in \mathfrak{R}(A)$ and by monotonicity of integral one has*

$$\left| \int_A f \right| \le \int_A |f| \,.$$

Corollary 8.3. *Monotonic function $f\colon [a,b] \to \mathbb{R}$ is integrable.*

Jordan-measurable Sets and Content

By Theorem 8, the characteristic function of a bounded set $C \subseteq \mathbb{R}^n$ is integrable if and only if the boundary of C has measure zero (since the set of discontinuous points is the boundary of C).

Definition (Jordan-measurable). A bounded set $C \subseteq \mathbb{R}^n$ whose boundary has measure zero is called **Jordan-measurable**. In that case, the n-dimensional volume (or n-dimensional content) of C is defined by

$$\mathrm{vol}^n(C) := \int_C 1 = \int_A \mathcal{X}_C,$$

where A is a closed rectangle contains C.

The basic properties of Jordan-measurable sets are listed as follows.

(i) If $C \subseteq \mathbb{R}^n$ is a finite set that C is Jordan-measurable and $\mathrm{vol}^n(C) = 0$;

(ii) Finite union/intersection of Jordan-measurable sets is also Jordan-measurable;

(iii) vol^n is finitely additive;

(iv) Even if bounded open/closed set may not be Jordan-measurable;

(v) For any Jordan-measurable $F \subseteq \mathbb{R}^n$, $\mathrm{vol}^n(F) = 0$ implies $m^*(F) = 0$;

(vi) For compact set F, $\mathrm{vol}^n(F) = 0$ if and only if $m^*(F) = 0$.

Proposition 9. *Let $F \subseteq \mathbb{R}^n$. The following two statements are equivalent.*

(i) *F is Jordan-measurable and $\mathrm{vol}^n(F) = 0$;*

(ii) *For any $\epsilon > 0$, there exists finitely many closed*[2] *rectangles $S_1, S_2, \ldots, S_k$ such that $F \subseteq \cup_{i=1}^k S_i$ and $\sum_{i=1}^k \mathrm{vol}^n(S_i) < \epsilon$.*

Proof. Suppose F is Jordan-measurable and $\int_A \mathcal{X}_F = 0$, where A is a closed rectangle containing F. For any given $\epsilon > 0$, choose a partition P such that $U(f, P) < \epsilon$. Then the collection of all subrectangles that intersect F is our desired covering.

Now, assume (ii) is true. Then F is Jordan-measurable (why?) and we want to prove $\int_A \mathcal{X}_F = 0$. Since $\underline{\int}_A \mathcal{X}_F \geq 0$ is evident, it remains to show $\overline{\int}_A \mathcal{X}_F \leq 0$. Let $\epsilon > 0$ be given, take $S_1, S_2, \ldots, S_k$ as in (ii). By expanding each S_i slightly if necessary, we can assume further that $F \subseteq \cup_{i=1}^k S_i^\circ$. Choose a partition P such that each subrectangle of P is either contained in some S_i or disjoint from F. For this partition P, one has

$$\overline{\int}_A \mathcal{X}_F \leq U(f, P) \leq \sum_{i=1}^k \mathrm{vol}^n(S_i) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the proof is completed. □

We know that if two functions differ only on a finite set then they have the same integrability and the same integral (in the case they are integrable). More generally, we have the following theorem.

Proposition 10. *Let $f, g\colon A \to \mathbb{R}$ be bounded functions (and A be a closed rectangle, as before). Suppose $f \equiv g$ on A except on a set E of content zero. Then f and g have the same integrability. If, in addition, they are integrable, then*

$$\int_A f = \int_A g.$$

Proof. By the linearity of integration, we can assume without loss of generality that $g \equiv 0$.

Let $M > 0$ be the bound of $|f|$ (i.e. $|f| \leq M$ on A) and let $\epsilon > 0$ be given. Since E has content zero, Theorem 9 guarantees that there exists finitely many closed rectangles $S_1, S_2, \ldots, S_k \subseteq A$ such that $E \subseteq \cup_{i=1}^k S_i^\circ$ and $\sum_{i=1}^k \mathrm{vol}^n(S_i) < \epsilon$. Choose a partition P such that each subrectangle of P is either contained in some S_i or disjoint from E. For this partition P, one has

$$-M\epsilon \leq L(f, P) \leq U(f, P) \leq M\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $f \in \mathfrak{R}(A)$ and $\int_A f = 0$. □

Exercise. Let $C \subseteq \mathbb{R}^n$ be Jordan-measurable. Prove that for any $\epsilon > 0$ there exists a compact set $K \subseteq C$ such that $\int_{C-K} 1 < \epsilon$.

[2]The word "closed" can be replaced by "open" in here.

Fubini's Theorem

Roughly speaking, Fubini's Theorem says that we can interchange the order of integrals. The precise statement is given as follows.

Theorem 11 (Fubini's Theorem). *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be closed rectangles, and let $f\colon A \times B \to \mathbb{R}$ be integrable. For each $x \in A$, define the function $g_x\colon B \to \mathbb{R}$ by $g_x(y) := f(x, y)$ and define also*

$$\mathfrak{L}(x) := \underline{\int_B} g_x = \underline{\int_B} f(x,y)\,dy, \qquad \mathfrak{U}(x) := \overline{\int_B} g_x = \overline{\int_B} f(x,y)\,dy.$$

Then $\mathfrak{L}, \mathfrak{U} \in \mathfrak{R}(A)$ and

$$\int_A \mathfrak{L} = \int_A \mathfrak{U} = \int_{A\times B} f.$$

Sometimes it is written as

$$\int_A \underline{\int_B} f(x,y)\,dydx = \int_A \overline{\int_B} f(x,y)\,dydx = \int_{A\times B} f.$$

Remark. Similarly we have

$$\int_B \underline{\int_A} f(x,y)\,dxdy = \int_B \overline{\int_A} f(x,y)\,dxdy = \int_{A\times B} f.$$

Therefore,

$$\int_A \underline{\int_B} f(x,y)\,dydx = \int_B \underline{\int_A} f(x,y)\,dxdy,$$

and

$$\int_A \overline{\int_B} f(x,y)\,dydx = \int_B \overline{\int_A} f(x,y)\,dxdy.$$

If f is continuous, then

$$\int_A \int_B f(x,y)\,dydx = \int_B \int_A f(x,y)\,dxdy,$$

saying that we can interchange the order of integrals.

Proof. Let P_A be a partition of A and P_B a partition of B. Together they give a partition P of $A \times B$ in the natural way. Note that

$$\begin{aligned} L(f,P) &= \sum_S m_S(f) \cdot \mathrm{vol}^n(S) \\ &= \sum_{S_A,S_B} m_{S_A \times S_B}(f) \cdot \mathrm{vol}^n(S_A \times S_B) \\ &= \sum_{S_A} \left(\left(\sum_{S_B} m_{S_A \times S_B}(f) \cdot \mathrm{vol}^n(S_B) \right) \mathrm{vol}^n(S_A) \right). \end{aligned}$$

Note also that, for each $x \in S_A$, one has

$$\begin{aligned} \sum_{S_B} m_{S_A \times S_B}(f) \cdot \mathrm{vol}^n(S_B) &\leq \sum_{S_B} m_{S_B}(g_x) \cdot \mathrm{vol}^n(S_B) \\ &\leq \mathfrak{L}(x). \end{aligned}$$

Therefore,

$$\begin{aligned} L(f,P) &\leq \sum_{S_A} \left(\left(\inf_{x \in S_A} \mathfrak{L}(x) \right) \mathrm{vol}^n(S_A) \right) \\ &= L(\mathfrak{L}, P_A). \end{aligned}$$

Similar consideration gives $U(f,P) \geq U(\mathfrak{U}, P_A)$. So, we have

$$L(f,P) \leq L(\mathfrak{L}, P_A) \leq U(\mathfrak{L}, P_A) \leq U(\mathfrak{U}, P_A) \leq U(f,P). \tag{10}$$

Since (10) is true for all P_A, P_B, and since f is assumed to be integrable, it follows that

$$\int_A \mathfrak{L} = \int_{A \times B} f.$$

The assertion for $\mathfrak{U}$ can be proved by similar argument. □

Here is an application of Fubini's Theorem.

Proposition 12. *Let $f \colon \mathbb{R}^n \to \mathbb{R}$ $(n \geq 2)$ be a function. If $D_1 D_2 f$, $D_2 D_1 f$ exist and are continuous, then $D_1 D_2 f = D_2 D_1 f$.*

Proof. Without loss of generality, we assume $n = 2$. Supposing the conclusion is false. By symmetry we can assume $D_1 D_2 f(c) > D_2 D_1 f(c)$ for some $c \in \mathbb{R}^2$, then the continuity of $D_1 D_2 f$, $D_2 D_1 f$ guarantees that there is a closed rectangle $A = [a_1, b_1] \times [a_2, b_2]$ containing c such that

$$D_1 D_2 f > D_2 D_1 f \qquad \text{on A.}$$

So, we have

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} D_1D_2f(x,y)\,dxdy > \int_{a_2}^{b_2}\int_{a_1}^{b_1} D_2D_1f(x,y)\,dxdy. \tag{11}$$

The left member of (11) can be easily calculated by the Fundamental Theorem of Calculus while the right member can be calculated using Fubini's Theorem.

$$\begin{aligned}
\text{L.H.S. of (11)} &= \int_{a_2}^{b_2} D_2f(b_1,y) - D_2f(a_1,y)\,dy \\
&= f(b_1,b_2) - f(b_1,a_2) - f(a_1,b_2) + f(a_1,a_2). \\
\text{R.H.S. of (11)} &= \int_{a_1}^{b_1}\int_{a_2}^{b_2} D_2D_1f(x,y)\,dydx \qquad \text{(by Fubini's Theorem)} \\
&= \int_{a_1}^{b_1} D_1f(x,b_2) - D_1f(x,a_2)\,dx \\
&= f(b_1,b_2) - f(a_1,b_2) - f(b_1,a_2) + f(a_1,a_2).
\end{aligned}$$

It turns out that (11) is a contradiction. □