Metric Space

Introduction

In $\mathbb{R}^2$, the distance between 2 points (x_1,y_1) and (x_2,y_2) is defined to be $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. The distance between 2 finite subsets of $\mathbb{R}^2$, A and B ,can be defined as $\max\limits_{a\in A,b\in B}|a-b|$. You may wonder if we have another way to define distance in the above two examples. Generally, if we are given a set X, can we define the distance between 2 elements in X?

Definition 1.1

For a set X(more than 1 element), suppose there is a function d, which maps X^2 to $\mathbb{R}^+\cup\{0\}$, and satisfy the following three properties:

For x, y and $z\in X$,

(i) $d(x,y)\geq 0$; $d(x,y)=0$ if and only if $x=y$;

(ii) $d(x,y)=d(y,x)$;

(iii) $d(x,y)\leq d(x,z)+d(y,z)$ (Triangle inequality)

Then (X,d) is called a metric space, and d is called distance function, or metric.

Examples

(a) For $x,y\in\mathbb{R}$, define $d(x,y)=|x-y|$. Then ($\mathbb{R}$ d) is a metric space. In general, for $x=(x_1,x_2,...,x_n)$ and $y=(y_1,y_2,...,y_n)\in\mathbb{R}^n$, define

$$d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2}\ .$$

Then $(\mathbb{R}^n,d)$ is a metric space. As this d is usually used, we called it the usual metric.

(b) For $x,y\in\mathbb{R}^n$, define $d(x,y)=\max\limits_{1\leq i\leq n}|x_i-y_i|$. Then $(\mathbb{R}^n,d)$ is a metric space.

(c) Let X be the set of all continuous function on [0,1], denoted by $C[0,1]$. Define $d(f,g)=\int_0^1|f(x)-g(x)|\,dx$ for $f,g\in X$. (X,d) is then a metric space. For this X, we can define $d'(f,g)=\max\limits_{x\in[0,1]}|f(x)-g(x)|$. (X,d') is also a metric space.

(d) Define $d(x,y)=0$ if $x=y$, and otherwise 1 in X. Then (X,d) is a metric space. d here is called the discrete metric.

Readers are suggested to verify the three properties (i),(ii) and (iii) of the above examples. We also see that for a set we can have more than a metric.

Definition 1.2

In the metric space (X,d), we said a sequence x_n converges to x, denoted by $x_n \to x$, if for every $\varepsilon > 0$, there exists an integer K such that $n>K$ implies $d(x_n, x) < \varepsilon$.

This is an analogue to what you have learnt about convergence in your first real analysis course.

Theorem 1.3

If x_n converges to x and y in (X,d), then $x=y$.

Proof. For $\varepsilon > 0$, there exists K such that $n>K$ implies $d(x_n, x) < \frac{\varepsilon}{2}$ and $d(x_n, y) < \frac{\varepsilon}{2}$. Then $d(x, y) \le d(x, x_n) + d(y, x_n) < \varepsilon$. Since ε is arbitrary, we have $d(x,y)=0$, hence $x=y$.

From this we know a convergent sequence has a unique limit.

Theorem 1.4

Suppose d_n is a sequence of metric of X. If, for every $x,y \in X$, $d_n(x,y)$ is bounded, and $d(x,y) = \limsup_{n\to\infty} d_n(x, y) \ne 0$ for $x \ne y$, then d is a metric on X. (Those who do not know limit supremum can ignore this theorem.)

Proof. The first two properties of d are trivial. To see the triangle inequality, note that for x, y and $z \in X$,

$$\limsup_{n\to\infty} d_n(x, y) \le \limsup_{n\to\infty} \{d_n(x, z) + d_n(z, y)\} \le \limsup_{n\to\infty} d_n(x, z) + \limsup_{n\to\infty} d_n(y, z).$$

This establishes the triangle inequality for d.

Corollary

Suppose d_n is a sequence of metric of X and $d(x, y) = \lim_{n\to\infty} d_n(x, y)$ exists for all $x,y \in X$ and nonzero for $x \ne y$, then d is a metric of X.

Proof. This follows easily from theorem 1.4.

Theorem 1.5

Suppose $x_n \to x$ in the metric space (X,d). Then for $\varepsilon > 0$, there exist an integer N such that for $n,m>N$, $d(x_n,x_m)<\varepsilon$.

Proof. Pick $\varepsilon > 0$ and choose N so that for $n>N$, $d(x_n,x)<\frac{\varepsilon}{2}$. Then for $n,m>N$,

$$d(x_m,x_n) \le d(x_m,x) + d(x_n,x) < \varepsilon$$

Exercise

1. If d_1 and d_2 are metrics of X, is it true that d_1+d_2, d_1-d_2, d_1*d_2, and $\sqrt{d_1}$ are metrics of X?
2. Let X be the space of all continuous function on [0,1], d and d' to be the metric as in (c) under definition 1.1. Show if f_n converges to f in (X,d'), then f_n converges to f in (X,d). Is the converse true?
3. We say that two metrics d and d' are equivalent on X if a convergent sequence in (X,d) is a convergent in (X,d'), and vice versa. Given a metric d of X, prove that there exits a metric d' equivalent to d and bounded by 1.
4. Suppose d_n is a sequence of metric of X. Show that there exists a metric d bounded by 1, such that the convergent sequence in (X,d) is a convergent in each (X,d_n).

Open sets

We have open intervals on the real line. Let us generalize this concept in metric space. In the entire chapter, we always assume (X,d) is a metric space.

Definition 2.1

For $x \in X$, define

$$B_d(x,r) = \{y \in X : d(x,y) < r\}.$$

$B_d(x,r)$ is called an open ball centered at x with radius r. Any open ball centered at x is called a neighborhood of x.

Definition 2.2

For a subset S of X, define the interior of S, denoted by int(S), to be the set of points which have a neighborhood lying inside S. To be precise, $x \in \text{int}(S)$ if and only if there exists $r>0$ such that $B_d(x,r) \subseteq S$.

Examples

(a) (1,3) in $\mathbb{R}$ is an open ball centered at 2 with radius 1. Indeed, every open interval is an open ball.

(b) $\{(x,y): x^2+y^2<1\}$ is an open ball centered at the origin with radius 1.

(c) $\text{int}([1,2])=(1,2)=\text{int}([1,2)) = \text{int}((1,2]) = \text{int}((1,2))$

(d) The interior of the set of integer is the empty set.

Definition 2.3

A set S is said to be open if int(S)=S, or equivalently for every $x \in S$, there exists r>0 such that $B_d(x,r) \subseteq S$.

Examples

(a) Every open ball is an open set.

(b) The interior of any set is open.

(c) Let X=C[0,1], and $d(f,g)=\max_{x\in[0,1]}|f(x)-g(x)|$. Then $\{f\in X: f>0\}$ is an open set.

(d) In a metric space with discrete metric, every element is an open set.

Theorem 2.4

(i) $\varnothing$, X are open.

(ii) Any union of open sets(maybe uncountable union) is open.

(iii) Finite intersection of open sets is open.

Proof. (i) is trivial from the definition of open. (ii) Let $S=\bigcup S_\alpha$, where S_α is open.(Here we use α to indicate that the union may not be countable.) For $x\in S$, $x\in S_\alpha$ for some α. Then x has a neighborhood lying in S_α, hence lying in S. This shows (ii). Let $S=\bigcap_{i=1}^{n} S_i$, where S_i is open. For $x\in S$, $x\in S_i$, and hence there exists r_i such that $B_d(x,r_i) \subseteq S_i$ for $1\le i\le n$. Take $r=\min_{1\le i\le n} r_i$>0. Then $B_d(x,r)\subseteq S_i$ for all i, hence a subset of S. This shows (iii).

Remark: In Topology course, we encounter sets without metric. Then we define open sets according to the three properties in theorem 2.4.

Definition 2.5

Suppose f is a function from X to Y with metric d_X and d_Y respectively. f is said to be continuous at $x\in X$ if for any $\varepsilon>0$, there exists $\delta>0$ such that $d_X(x,y)<\delta$ implies $d_Y(f(x),f(y))<\varepsilon$.

This definition is analogue to what we have learnt about continuity in a real analysis course.

Theorem 2.6

$f: X\to Y$ is continuous at x if and only if $f(x_n)\to f(x)$ for any $x_n\to x$.

Proof. Suppose f is continuous at x and take $x_n\to x$. Then for $\varepsilon>0$, there exists $\delta>0$ such that $d_X(x,y)<\delta$ implies $d_Y(f(x),f(y))<\varepsilon$. Now we take N such that $n>N$ implies $d_X(x,x_n)<\delta$, and hence $d_Y(f(x),f(x_n))<\varepsilon$. Conversely, assume f is not continuous at x. Then there is $\varepsilon>0$ such that for each n we can have x_n such that $d_X(x_n,x)<\frac{1}{n}$ but $d_Y(f(x),f(x_n)\ge\varepsilon$. Now $x_n\to x$ but $f(x_n)$ does not converge to $f(x)$, a contradiction.

This theorem is an analogue to the sequential continuity theorem in the first real analysis course. The two proofs need exactly the same idea, except that the absolute sign is generalized to the metric. However, the following theorem may not be appeared so familiar.

Theorem 2.7

$f: X \to Y$ is continuous at every point of X if and only if $f^{-1}(U)$ is open for any open set U in Y. Recall that $f^{-1}(U) = \{x: f(x) \in U\}$

Proof. Suppose f is continuous at every point of X. For an open U and $x \in f^{-1}(U)$, there exists $\varepsilon > 0$ such that $B_{d_Y}(f(x), \varepsilon) \subseteq U$. The continuity at x shows there exists $\delta > 0$ such that

$f(B_{d_X}(x,\delta)) \subseteq B_{d_Y}(f(x),\varepsilon) \subseteq U$. This shows $B_{d_X}(x,\delta) \subseteq f^{-1}(U)$, and hence $f^{-1}(U)$ is open.

Conversely, for $x \in X$ and $\varepsilon > 0$, $f^{-1}(B_{d_Y}(f(x),\varepsilon))$ is open. Hence there exists $\delta > 0$ such that

$B_{d_X}(x,\delta) \subseteq f^{-1}(B_{d_Y}(f(x),\varepsilon))$. Hence we get the continuity of f at x.

Remark: In topology course, for X and Y without metric, the definition of continuity bases on open set, which is motivated by Theorem 2.6. We define $f: X \to Y$ to be continuous if $f^{-1}(U)$ is open for any open set U in Y.

Exercise

1. Let X be the space of all continuous function on [0,1], d and d' to be the metric as in (c) under definition 1.1. Show an open set in (X,d) is open in (X,d'), but the converse does not hold.
2. Does there exists a metric such that the open sets of X are precisely X and $\varnothing$?
3. Give an example that the identity map of a set X onto itself is not continuous.
4. Does there exists a metric d on X such that any mapping from X to Y is continuous on X?
5. Show that for any function f on $\mathbb{R}$, the set $\{x: f \text{ is continuous at } x\}$ is the countable intersection of open set.

Closed Sets

As expected, it is now time for the closed sets to take turn. Closed sets indeed have a lot to do with open sets.

Definition 3.1

A point x is said to be the limit point of a set S if there exists a sequence x_n which converges to x, where $x_n \in S$ and $x_n \neq x$.

Examples

(a) $[0,1)$ in $\mathbb{R}$ has limit point $\{0\}$ and $\{1\}$.

(b) $\{(x,y): x^2+y^2<1\}$ in $\mathbb{R}^2$ with usual metric has limit points $\{(x,y): x^2+y^2=1\}$

(c) The limit point of the integer set is the empty set.

(d) The limit points of the rational number in $\mathbb{R}$ are the whole real line.

Theorem 3.2

A point x is said to be a limit point of the set S if and only if for any open U containing x, U also contains $y \in S$ and $y \neq x$.

Remark: In the proof we write $B(x,r)$ instead of $B_d(x,r)$ as there is no confusion of the metric. We would include the subscript d in $B_d(x,r)$ only when we need to specify which metric are being used in latter text.

Proof. Suppose x is a limit point of S. For an open U which contains x, we have $r>0$ such that $B(x,r) \subseteq U$. Since from the hypothesis there is a sequence x_n in S converging to x, there exists for some m $x_m \neq x$ and $x_m \in B(x,r)$. This $y=x_m$ is the point we want. Conversely, for every n, $B(x,1/n)$ contains $x_n \neq x$ and $x_n \in S$. This x_n converges to x, which shows x is a limit point of S.

Remark: By this theorem, we can define limit points by open sets. This is the way we handle limit points in topology course.

Definition 3.3

The set S is said to be closed if S contains all its limit points.

Examples

(a) All closed intervals in $\mathbb{R}$ are closed sets.

(b) $\{(x,y): x^2+y^2 \leq 1\}$ in $\mathbb{R}^2$ with usual metric is closed.

(c) Any set consists of finite points is close.

Up to now we still do not see clearly how closed sets are related to open sets. The following theorem tells.

Theorem 3.4

S is closed if and only if $X \setminus S$, the complement of S, is open.

Proof. If S is closed, take x in its complement. Then x is not a limit point to S, hence there is a neighborhood of x lying outside S. This shows $X \setminus S$ is open. Conversely, let x be a limit point of S. Then for any open ball containing x intersects S, hence x is not an interior point of $X \setminus S$, and $x \in S$. This shows S is closed.

Equivalently, S is open if and only if $X \setminus S$, the complement of S, is closed. Therefore, we can define closed sets by open sets only.

Theorem 3.5(Compared to theorem 2.4)

(i) $\varnothing$, X are closed.

(ii) Any intersection of closed sets(maybe uncountable union) is closed.

(iii) Finite union of closed sets is closed.

Proof. (i) is trivial. To see (ii), let $F = \bigcap F_\alpha$, where F_α is closed. Write $F_\alpha = X \setminus U_\alpha$ where U_α is open. Then $F = \bigcap X \setminus U_\alpha = X \setminus \bigcup U_\alpha$ is closed by theorem 2.4 and 3.4. (iii) follows exactly the same way: $F = \bigcup_{i=1}^{n} F_i = X \setminus \bigcap_{i=1}^{n} U_i$, which is close.

Parallel to theorem 2.6, we have the same relation between continuity and closed sets.

Theorem 3.6

$f : X \to Y$ is continuous at every point of X if and only if $f^{1}(F)$ is close for any closed set F in Y.

Proof. By theorem 2.6, we have for any closed F, f is continuous on X $\Leftrightarrow$ $f^{1}(Y \setminus F) = X \setminus f^{-1}(F)$ is open $\Leftrightarrow$ $f^{1}(F)$ is closed.

Given an arbitrary set S, can we induce a closed set which contains S?

Definition 3.7

For a set S, define the closure of S, denoted by $\overline{S}$, to be the union of S and limit points of S.

It may not be so clear that for any set S, $\overline{S}$ is a closed set which contains S. Indeed, from the next theorem, $\overline{S}$ is the "smallest" close set containing S.

Theorem 3.8

For the set S, define C={F: F is close and contains S}, a collection of closed sets. Then $\overline{S} = \bigcap_{F \in C} F$, the intersection of all elements of C.

Proof. First it is easy to see S is a subset of the closed set $\bigcap_{F\in C} F$. If x is a limit point of S, x is a limit point of F, and hence in F. This shows $\overline{S} \subseteq \bigcap_{F\in C} F$. Suppose there is a point y in $\bigcap_{F\in C} F$ but not in $\overline{S}$. Then there exists $r>0$ such that $B(y,r)$ lies in the complement of S. Note that the close set $X \setminus B(y,r)$ contains S is in the collection C. Since $X \setminus B(y,r)$ does not contain y, so as $\bigcap_{F\in C} F$, which yields a contradiction. Therefore $\bigcap_{F\in C} F \subseteq \overline{S}$, the proof is completed.

Corollary

S is closed if and only if $\overline{S}=S$.

Proof. Obviously, $S \subseteq \overline{S}$. If S is close, S is in the collection C. Hence $\overline{S} = \bigcap_{F\in C} F \subseteq S$. The converse is trivial.

From the above, we see that for any close F containing S, $\overline{S}$ is a subset of F. $\overline{S}$ is the "smallest" in this sense.

Exercise

1. What is the closure of the rational numbers in $\mathbb{R}$?
2. Let X and Y be two metric spaces with the same metric d and $X\subseteq Y$. If U is open in X, must it be open in Y? How about the case of close?
3. Let $X=C[0,1]$, d and d' to be the metric as in (c) under definition 1.1. Is the set $\{f : f \geq 0\}$ close in (X,d)? How about in (X,d')?
4. Show F is close in X if and only if there is a continuous $f : X \rightarrow \mathbb{R}$ such that $f^{-1}(0)=F$.
5. If F and H are two disjoint sets, where F and H are close. Show that there exists two disjoint open sets which contains F and H respectively.

Compact sets

We define compact set to be the closed and bounded set in $\mathbb{R}^n$. However, the definition of compact is not exactly the same in metric space.

Definition 4.1

The set S is compact if any infinite sequence in S has convergent subsequence in S.

This definition is equivalent to close and bounded in $\mathbb{R}^n$, by Balzono-Wiestrass Theorem. From the definition it is very easy to see that a compact set is close in any metric space. However, in general, close and bounded does not imply compact.

Wee can see that we need the concept of convergence(hence metric) in the definition of 4.1. Can we base the definition on open sets only as in defining the close sets and continuity? Theorem 4.3 will answer. Before going to that, let us have one more definition first.

Definition 4.2

Let *C* be a collection of some open sets. If $S \subseteq \bigcup_{U \in C} U$, then we say that *C* is an open cover of *S*. *C* is called a finite cover of *S* if *C* is an open cover of *S* with finitely many elements. *D* is called a subcover of *S* with respect to *C* if $D \subseteq C$ and $S \subseteq \bigcup_{U \in D} U$.

For example, $C = \{(\frac{1}{n}, 1) : n \in N\}$ is an open cover of (0,1) but not [0,1]. If we add the intervals (-0.1,0.1) and (0.9,1.1) to *C*, then *C* is an open cover of [0,1].

Theorem 4.3

S is compact if and only if every open cover of *S* has a finite subcover.

The second part of the theorem means that for an open cover *C* of *S*, there exists $D \subseteq C$ with finitely many elements, where *D* is a cover of *S*. Here *C* may have uncountably many elements.

Proof. Suppose *S* is compact and there is an open cover *C* which cannot have a finite subcover. Exclude those elements in *C*, which is a subset of the union of other elements. Therefore for any $U \in C$, we can choose $x \in C$ such that if $U \neq V \in C$, $x \notin V$. Now for each U_i, pick x_i in such a manner. As no finite cover with respect to *C* exists, $\{x_i\}$ forms an infinitely sequence. Hence there is a subsequence converging to *x* in *S*. Now $x \in U$, for some $U \in C$. This *U* contains infinitely many x_i, contradicting the choice of x_i.

Conversely, suppose there is an infinite sequence $\{x_i\}$ which does not have convergent subsequence. Then for each $x \in S$ and $x \neq x_i$ for any *i*, there is $r_x > 0$ such that $B(x,r_x)$ contains no x_i, or if $x = x_i$, there is $r_x > 0$ such that $B(x,r_x)$ contain one x_i only.(Why?) Now $C = \{B(x,r_x): x \in S\}$ is an open cover of *S*, which does not have finite subcover, a contradiction. This completes the proof.

Theorem 4.3

Any close subset *S* of a compact set *K* is compact.

First proof. For an infinite sequence $\{x_n\}$ in *S*, it has a subsequence converging to *x* in *K*. Then *x* is a limit point of *S*, hence in *S*. This proves the compactness of *S*.

Second proof. Suppose $S \subseteq \bigcup U_\alpha$, where U_α is open. Let $U = X \setminus S$, which is open. Then U_α and *U* together form an open cover of *K*. Since *K* is compact, from this open cover there exists a finite subcover, which means $K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \cup U$. Hence $S \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. We have found a finite subcover of S.

Remark: The second proof does not involve metric. It will be the proof of theorem 4.3 in topological spaces.

Corollary
Any intersection(including uncountable) of compact sets is compact.

Theorem 4.4
Finite union of compact sets is compact.

Proof. Let $K = \bigcup_{i=1}^{n} K_i$, where K_i is compact, and C be an open cover of K. Now for each K_i, we pick finite elements of C to form a subcover C_i. Now $\bigcup_{i=1}^{n} C_i$ is a finite subcover of K.

Now we turn to see the relation between compact sets and continuity.

Theorem 4.5
If $f : X \to Y$ is continuous on X, then $f(K)$ is compact in Y for any compact subset K of X.
Recall that $f(K)=\{f(x): x \in K\}$

Proof. Take any infinite sequence $\{f(x_n)\}$ in $f(K)$. Then there is a subsequence $x_{n_j} \to x \in K$. This implies $f(x_{n_j}) \to f(x)$ in $f(K)$.

Readers are suggested to construct a proof of theorem 4.5 without involving metric as an exercise. Now we turn to see the generalized extreme value theorem.

Theorem 4.6
Suppose $f : X \to Y$ is continuous on K compact in X, and $y \in Y$. Then there exists $x \in K$ such that $d_Y(f(x),y) \geq d_Y(f(z),y)$ for all $z \in K$.

Proof. First we show that there exists M such that $d_Y(f(z),y) \leq M$ for all $z \in K$. Otherwise, for each positive integer n we have z_n in K so that $d_Y(f(z_n),y) \geq n$. Since K is compact, $\{z_n\}$ has a subsequence converging to z. Then by the continuity of f, $d_Y(f(z),y)>n$ for all n, a contradiction.

Next, we prove the existence of x. As $d_Y(f(z),y)$ is bounded on K, $\sup_{z \in K} d_Y(f(z), y) = M$ is a real number. By supremum property, there exists x_n such that $d_Y(f(x_n),y) \to M$. $\{x_n\}$ has a subsequence converging to x, and by the continuity of f, $f(x)=M$.

Remark: Actually we do not need to assume X to be a metric space in theorem 4.6. We can proceed the proof by only considering open sets in X. (See exercise 4)

The extreme value theorem we learnt in high school or the first analysis course is just the special case of this theorem, when we take $X=Y=\mathbb{R}$ (the real set), K to be a closed and bounded interval, d to be the usual metric and $y=0$.

Before we end this section, let us go a little bit deeper to the concept of uniform continuity. Some of the beginning analysis course includes this concept, and some does not. Anyway, let us see the definition first.

Definition 4.7

A function $f : X \to Y$ is said to be uniformly continuous on a set S if for every $\varepsilon>0$, there exists $\delta>0$ such that $d_Y(f(x_1),f(x_2))< \varepsilon$ whenever $d_X(x_1,x_2)< \delta$ for x_1 and x_2 in S.

Obviously uniformly continuous are continuous. In general, continuity does not imply uniform continuity. However, in some conditions, the weaker one implies the stronger.

Theorem 4.8

If $f : X \to Y$ is continuous on a compact K, then f is uniform continuous on K.

Proof. Suppose it is not the case. Then there exists $\varepsilon>0$, such that for each n we can pick a_n and b_n in K, where $d_X(a_n,b_n)<\frac{1}{n}$ but $d_Y(f(a_n),f(b_n))\geq\varepsilon$. Now $\{a_n\}$ has a subsequence $a_{n_j} \to a$ in K. Thus

$b_{n_j} \to a$ and $f(b_{n_j}) \to f(a)$. To reach a contradiction, choose N so that $j>N$ implies

$d_Y(f(a_{n_j}), f(a)) < \frac{\varepsilon}{2}$. Then

$$\varepsilon \leq d_Y(f(b_{n_j}), f(a_{n_j})) < d_Y(f(b_{n_j}), f(a)) + d_Y(f(a), f(a_{n_j}) < d_Y(f(b_{n_j}), f(a)) + \frac{\varepsilon}{2}$$

for all $j>N$. This shows $\frac{\varepsilon}{2} \leq d_Y(f(b_{n_j}), f(a))$, which is wrong.

Exercise

1. Suppose there is an open cover C of a compact K. Show there exists $r>0$ such that for any $x\in K$, there exists $U\in C$ and $B(x,r)\subseteq U$.
2. Let X be the unit circle in $\mathbb{R}^2$ with the usual metric. Does there exists a continuous function which maps X onto the real line?

3. Construct a close and bounded set which is not compact in a metric space (X,d).
4. Suppose $f: X \to Y$ is continuous on X, K is compact and $y \in Y$.
   (i) Show $g: X \to \mathbb{R}$, defined by $g(x)=d_Y(f(x),y)$ is continuous by showing $g^{-1}(U)$ is open for any open set U in $\mathbb{R}$.
   (ii) Show that $d_Y(f(x),y)$ is bounded for $x \in K$, by considering the open sets $[0,n)$ on $\mathbb{R}^+ \cup \{0\}$.
   (iii) Let $M = \sup\{d_Y(f(x),y): x \in K\}$. Show there exists $x \in K$ such that $f(x)=M$ by considering the open sets $[0, M-\frac{1}{n})$.
5. Suppose X is a metric space such that every closed ball is compact. If there is an open U which contains a compact K, then there exists another compact H such that $K \subseteq H \subseteq U$

Complete sets (additional topic)

Definition 5.1

A sequence $\{x_n\}$ is set to be a Cauchy sequence if for $\varepsilon>0$, there exists a positive integer N so that $n,m>N$ implies $d(x_n,x_m)<\varepsilon$.

Noted that from theorem 1.5 every convergent sequence is a Cauchy sequence. In the real set $\mathbb{R}$, we know that a Cauchy sequence is the same as a convergent sequence. However it is always not the case. For example, take $X=Q$, the rational number. Then a Cauchy sequence is not a convergent sequence anymore, since rational sequence may converge to an irrational number, which is not in X. This leads to the following definition.

Definition 5.2

The set S is said to be complete if all the Cauchy sequence in S is a convergent sequence.

Clearly, a compact set is complete. A close subset of a complete set is complete. Let us see an interesting theorem of complete metric spaces.

Theorem 5.3(Contractive mapping theorem)

Suppose $f: X \to X$, where X is complete, has the property that $d(f(x_1),f(x_2)) \le cd(x_1,x_2)$ for all x_1, x_2 in X, where $c<1$. Then f is continuous on X, and there exists unique x so that $f(x)=x$.

Proof. The continuity of f is trivial. Now pick $x_1 \in X$, and define $x_{n+1}=f(x_n)$. Then

$$d(x_n,x_{n-1}) = d(f(x_{n-1}), f(x_{n-2})) \le cd(x_{n-1},x_{n-2}) \le c^{n-2}d(x_2,x_1),$$

and for $m>n$,

$$d(x_m,x_n) \le \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \le \sum_{i=n}^{\infty} d(x_i,x_{i+1}) \le \sum_{i=n}^{\infty} c^{i-1}d(x_1,x_2) \le \frac{c^{n-1}}{1-c}d(x_1,x_2).$$

As the last inequality tends to 0 when $n \to \infty$, we know $\{x_n\}$ is a Cauchy sequence and hence x_n converges to $x \in X$. By the continuity of f, we have $x=f(x)$.

The uniqueness of the fixed point x can be seen by supposing there is another fixed point y, and $d(x,y)=d(f(x),f(y)) \leq cd(x,y)$. This forces $d(x,y)=0$, as $c<1$.

This theorem can be seen in such an interesting way: When you are using a computer and reduce a maximized window to a smaller size, a unique point on the screen is fixed!

We are now going to reach a more advanced theorem: the Baire’s Theorem. Before reaching that, let us go over the dense sets first.

Definition 5.4

Suppose $S \subseteq T$ in the metric space X. Then S is dense in T if for $x \in T$ and $\varepsilon >0$, there exists $y \in S$ such that $d(x,y)< \varepsilon$

For example, the rational set is dense in $\mathbb{R}$. As every point of T is a limit point of the dense subset S, we see that $\overline{S} = \overline{T}$. Conversely, if $S \subseteq T$ and $\overline{S} = \overline{T}$, then S is dense in T. It is also not difficult to see that if S is dense in T and T is dense in U, then S is dense in U.

Now let us turn to the concept of nowhere dense sets.

Definition 5.5

The set S is said to be nowhere dense if any open set contains an open ball that is disjoint with S.

For example, a finite set and the integer set in $\mathbb{R}$ are nowhere dense. Any open set and the rational set in $\mathbb{R}$ are not nowhere dense. It is easy to see that if a set is nowhere dense, so are its subsets.

Theorem 5.6

S is nowhere dense if and only if $\overline{S}$ is nowhere dense.

Proof. Suppose S is nowhere dense. An open U contains an open ball B which is disjoint to S. Then no point of B is the limit point of S, since otherwise B would contain some points of S. Hence $\overline{S}$ is disjoint to B. This shows $\overline{S}$ is nowhere dense. The converse is trivial.

Now we reach the last theorem of this section.

Theorem 5.7(the Baire’s Theorem)

Suppose U_i is open dense in a complete metric space X for all integer i. Then $\bigcap_{i=1}^{\infty} U_i$ is dense in X.

Proof. Let $y \in X$ and $\varepsilon > 0$. We need to show there is $x \in \bigcap_{i=1}^{\infty} U_i$ such that $B(x,\varepsilon)$ contains y. As U_1 is dense, there exists $x_1 \in U_1$ such that $B(x_1, \frac{\varepsilon}{2})$ contains y. Also we can choose a $r_1 < \frac{\varepsilon}{2}$ so that $\overline{B(x_1,r_1)}$ is a subset of the open U_1. Then for each $i>1$, there exists $x_i \in U_i$ and $r_i>0$ so that $B(x_i, \frac{r_{i-1}}{2})$ contains x_{i-1} and $\overline{B(x_i,r_i)}$ is a subset of U_i with $r_i < \frac{r_{i-1}}{2}$. By the choice of r_i we have $B(x_i,r_i) \subseteq B(x_{i-1},r_{i-1})$. Now for $m>n$,

$$d(x_m,x_n) \le \sum_{i=n}^{m-1} d(x_{i+1},x_i) \le \sum_{i=n}^{\infty} d(x_{i+1},x_i) \le \sum_{i=n}^{\infty} \frac{r_i}{2} \le \frac{\varepsilon}{2^n}.$$

The last inequality tends to 0 when $n \to \infty$. This shows $\{x_n\}$ is a Cauchy sequence, hence the completeness of X gives the existence of the limit x. As $\{x_i\}_{i=n}^{\infty}$ is a sequence in the close set $\overline{B(x_n,r_n)}$, so $x \in \overline{B(x_n,r_n)}$. Thus $x \in U_n$ for all n and of course in the set $\bigcap_{i=1}^{\infty} U_i$. Finally, as $x \in \overline{B(x_1,r_1)}$, we have

$$d(x,y) \le d(x,x_1) + d(x_1,y) < \varepsilon.$$

We see then $B(x,\varepsilon)$ contains y and the proof is completed.

Corollary

The complete metric space X cannot be the countable union of nowhere dense sets.

Proof. Let $X = \bigcup_{i=1}^{\infty} S_i$, where S_i is nowhere dense. It is no doubt that $X = \bigcup_{i=1}^{\infty} \overline{S_i}$. Note that $X \setminus \overline{S_i}$ is open dense. Now taking complement of both sides gives $\varnothing = \bigcap_{i=1}^{\infty} X \setminus \overline{S_i}$. However by theorem 5.7, the right side of the equality is dense, hence cannot be empty.

Exercise.

1. Suppose F_n is nonempty closed in a complete metric space X for all n and $F_1 \supseteq F_2 \supseteq \dots$. Define $d_n = \sup\{d(x,y) : x, y \in F_n\}$. If $d_n \to 0$, shows $\bigcap_{n=1}^{\infty} F_n$ is nonempty. Is it true that if X is not complete?
2. (1979 UC Berkeley PhD preliminary examination) An accurate map of California is spread out flat on a table in Evans Hall, in Berkeley. Prove that there is exactly one point on the map lying directly over the point it represents.
3. Let X to be the space of continuous on $\mathbb{R}$ with metric $d(f,g) = \sup_{x \in R} |f-g|$. Show that X is a

complete metric space.

4. Show that there does not exist a function $f:\mathbb{R}\to\mathbb{R}$ which is continuous on exactly the rational number but not the irrational. (Hint: Use Baire's Theorem and the result of exercise 5 in section 2.)
5. (i) Show that the Cantor set K is nowhere dense.

   (ii) Define $K_q=\{qK\colon q\in\mathbb{Q}\}$. Is the set $\bigcup_{q\in\mathbb{Q}} K_q$ close? Is it Open?

6. Solution.

Section 1

1. If d_1 and d_2 are metrics, then d_1+d_2 and $\sqrt{d_1}$ are metrics. In general, d_1-d_2 and $d_1\cdot d_2$ may not be a metric. For example, if we take $d_1=d_2=$ standard metric on the real line, then $d_1\cdot d_2=d_1^{\,2}$ is not a metric because the triangle inequality is not satisfied, for say $(d_1\cdot d_2)(0,2)=2^2=4$ while
$$(d_1\cdot d_2)(0,1)+(d_1\cdot d_2)(1,2)=1^2+1^2=2<4.$$
   Also d_1-d_2 is not a metric because it may not even be always non-negative (e.g. take $d_2=2d_1$).
2. If f_n converge to f in d', then
$$d\left(f_n,f\right)=\int_0^1\left|f_n(x)-f(x)\right|dx\le\int_0^1 d\left(f_n,f\right)dx=d\left(f_n,f\right)\to 0$$
   as n tends to infinity. This proves convergence in d' implies convergence in d. The converse is in general not true. For example, if we take $f_n(x)=x^n$ and $f(x)=0$, then f_n converges to f in d,

   but $d'\left(f_n,f\right)=1$ for all n.
3. Simply define d' by $d'(x,y)=\dfrac{d(x,y)}{1+d(x,y)}$ and check that d' defines a metric which is equivalent to d and bounded by 1.
4. Define d by $d(x,y)=\sum_{n=1}^{\infty}2^{-n}\dfrac{d_n(x,y)}{1+d_n(x,y)}$ and apply Question 3 above.

Section 2

1. It suffices to show that an open ball in (X,d) is open in (X,d').

   Let f be an arbitrary element in $B_d(g,r)$, let $\in=d(f,g)$,

   We claim that $B_{d'}(f,r-\varepsilon)\subset B_d(g,r)$, so f is an interior point of $B_d(g,r)$ with respect to the metric *d'* .Hence $B_d(g,r)$ is open in (*X*,*d*).

To prove the claim, let $h \in B_{d'}(f, r-\varepsilon)$, then *d*(*g*,*h*) =

$\int_0^1 | g(x)-h(x) |dx \le \int_0^1 | g(x)-f(x) | dx + \int_0^1 | f(x)-h(x) |dx$ <*r*. The result follows.

The converse is false. For $0 \in B_{d'}(0,r)$ and ε >0, consider the function g:$[0,1] \to \mathbb{R}$ defined by *g(x)*=0 for $x \in [\frac{\varepsilon}{2r}, 1]$, g(0)=0, and *g* is linear elsewhere. It can be checked that *g* is in $B_d(0,\varepsilon)$ but not in $B_{d'}(0,r)$. Thus $B_d(0,\varepsilon)$ is not contained in $B_{d'}(0,r)$ for any ε. That is, 0 is not an interior point of $B_{d'}(0,r)$ in (*X*,*d*).

2. No. suppose so, for $x,y \in X$, we have $y \in B(x,r) = X$ for any *r*>0. taking $r \to 0$,
   We have *x*=*y*, a contradiction.
3. Let *d* be the usual metric on $\mathbb{R}$, *d'* be the discrete metric on $\mathbb{R}$.
   Then $\frac{1}{n} \to 0$ in ($\mathbb{R}$,*d*) but not in ($\mathbb{R}$,*d'*). This shows the discontinuity of the identity map from ($\mathbb{R}$,*d*) to ($\mathbb{R}$,*d'*).
4. Yes, the discrete metric.
5. Let *S*={*x*: *f* is continuous at *x*}.For each $x \in S$ and positive integer *n* ,there exists $\delta_n(x)$ >0 such that *f*(*y*)$\in B(x,\frac{1}{n})$ whenever $y \in B(x,\delta_n(x))$ by the definition of continuity. Now let
   $\mathbb{R}$ .Clearly, T_n is open . We want to show that *S*=$\bigcap_{n=1}^{\infty} T_n$. $S \subseteq \bigcap_{n=1}^{\infty} T_n$ since *S* is contained in each T_n. To show $S \supseteq \bigcap_{n=1}^{\infty} T_n$, pick any *y* in $\bigcap_{n=1}^{\infty} T_n$. For ε >0, choose n_0 such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$ As $y \in T_{n_0}$, $y \in B(x,\delta_{n_0}(x))$ for some $x \in S$. Choose δ so that $B(y,\delta) \subseteq B(x,\delta_{n_0}(x))$. Then for $z \in B(y,\delta)$, $d(f(z),f(y)) \le d(f(z),f(x)) + d(f(x),f(y)) < \varepsilon$. Hence $y \in S$.

Section 3

1. $\overline{\mathbb{Q}} = \mathbb{R}$ by the density of $\mathbb{Q}$ in $\mathbb{Q} = \bigcap_{n=1}^{\infty} V_n$.
2. Both statements are false. In the first case, take *X*=[0,1], *Y*=$\mathbb{R}$. Then *X* is open in *X*, but not open in *Y*. For the converse, take *X*=(0,1), *Y*=$\mathbb{R}$. Then *X* is closed in *X* ,but not closed *Y*.
3. The answers are affirmative in both cases. Suppose $f_n \ge 0$ and $f_n \to f$ in (*X*,*d*). Suppose *f(x)*<0 for some $x \in [0,1]$. Take δ >0 so that *f*(*y*)<$\frac{f(x)}{2}$ for y$\in B(x,\delta)$.

Then $d(f_n, f) = \int_0^1 |f_n(y) - f(y)|dy \geq \int_{x-\delta}^{x+\delta} |f_n(y) - f(y)|dy > |f(x)|\delta$ for all *n*,a contradiction to the hypothesis that $f_n \to f$. So $f \geq 0$.

In (*X,d'*), Suppose $f_n \geq 0$ and $f_n \to f$. Then for each $x \in [0,1]$, $f_n(x) \to f(x)$ in the usual sense. Thus $f(x) \geq 0$.

4. For the "$\Leftarrow$"case, Note that $\{0\}$is closed in $\mathbb{R}$ implies $f^{-1}(0)$ is closed in *X* by Theorem 3.6.

For the "$\Rightarrow$"case, suppose *F* is closed in *X*, Define a function f:X$\to \mathbb{R}$ by f(x)$= \inf_{y \in F} d(x,y)$.

Clearly, $f^{-1}(0) = F$. It remains to prove that *f* is continuous at *x* for any $x \in X$. which immediately follows when we apply triangular inequality to prove that

$f(B(x, \frac{\varepsilon}{2})) \subset B(f(x), \varepsilon)$.

5. For $x \in F$, there exists r_x such that $B(x,r_x)$ is disjoint to *H*. Similarly, we can get $B(y,r_y)$ disjoint to *F* for $y \in H$. Define $U = \bigcup_{x \in F} B(x, \frac{r_x}{3})$ and $V = \bigcup_{y \in H} B(y, \frac{r_y}{3})$. If there is *z* in both *U* and *V*, *z* is in $B(x, \frac{r_x}{3})$ and $B(y, \frac{r_y}{3})$ for some $x \in F$ and $y \in H$. However, $\max(r_x, r_y) \leq d(x,y) \leq d(x,z) + d(y,z) \leq \frac{r_x + r_y}{3} \leq \frac{2}{3}\max(r_x, r_y)$, a contradiction.

Section 4

1. For each *x* in *K*, there exists an open set $U_x \in C$ and a real number $r_x > 0$ such that $B(x, r_x) \subseteq U_x$. Then $\{B(x, r_x) : x \in K\}$ is an open cover of *K*, so by compactness of *K* there exists finitely many points $x_1, x_2, ..., x_n$ in *K* such that $K \subseteq \bigcup_{i=1}^{n} B\left(x_i, r_{x_i}\right)$. Take $r = \min\left\{r_{x_i} : i = 1, 2, ..., n\right\} > 0$, then for each *x* in *K*, if we take an index *i* such that $x \in B\left(x_i, r_{x_i}\right)$, we will have, from $r \leq r_{x_i}$, that

$$B(x,r) \subseteq B\left(x_i, 2r_{x_i}\right) \subseteq U_{x_i} \in C.$$

(Note our choice of *r* is independent of our *x* in *K*.) Alternatively one can argue by contradiction.

If our assertion is false, then for each positive integer j there exists $x_j \in K$ such that $B\left(x_j, j^{-1}\right)$ is not subset of any $U \in C$. Since K is compact, the sequence $\left\{x_j\right\}$ has a convergent subsequence $\left\{x_{j_k}\right\}$ with limit x in K. Now for this x there exists $U \in C$ and $r > 0$ such that $B(x,r) \subseteq U$. For k sufficiently large, since $\lim_{k\to\infty} x_{j_k} = x$ and $\lim_{k\to\infty} {j_k}^{-1} = 0$, we have $B\left(x_{j_k}, {j_k}^{-1}\right) \subseteq B(x,r) \subseteq U$. This contradicts our choice of x_j.

2. No, because the unit circle is compact but the real line is not, while continuous functions preserve compactness.
3. Consider an infinite set X with the discrete metric $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$. Then X is closed and bounded, but it is not compact, because a sequence in X whose all terms are distinct has no convergent subsequence. Alternatively, a less artificial example is the sequence l^2 space, which consists of all sequences $\{x_n\}$ for which $\sum |x_n|^2 < \infty$. If, for each positive integer n, we define y_n to be the sequence in the l^2 space for which the n-th entry is 1 and all other entries are zero, then the set of all y_n's is a closed and bounded set under the l^2 metric and yet it is not compact (because the sequence $\{y_1, y_2, y_3, ...\}$ in l^2 has no convergent subsequence).
4. (i) Simply note that $g^{-1}((a,b))=f^{-1}(S)$ where $S=\{z: a<d_Y(z,y)<b\}$. As S is open, $g^{-1}((a,b))$ is open. The case is the same for the inverse image of $[0,r)$. Hence the continuity of g follows from that of f.

   (ii) Note that $\bigcup_{n=1}^{\infty} g^{-1}([0,n))$ covers K hence by the compactness of K, K can be covered by $\bigcup_{i=1}^{k} g^{-1}([0,n_i))$ for some k. Hence $d_Y(f(x),y)$ is bounded by the maximum among n_i.

   (iii) Otherwise $\bigcup_{n=1}^{\infty} g^{-1}([0, M - \frac{1}{n}))$ forms an open cover of K, but does not exists a finite subcover.
5. For each point x in K there exists an open ball B_x centered at x such that its closure $\overline{B_x} \subseteq U$. The family of all B_x, where x runs through the whole K, is an open cover of K. By compactness of K there exists a finite subcover $\{B_{x_1}, B_{x_2}, ..., B_{x_n}\}$ of K. Now each of the $\overline{B_{x_i}}$ is a closed ball

and hence compact. Hence if we set $H = \bigcup_{i=1}^{n} \overline{B_{x_i}}$, then *H*, being a finite union of compact sets, is compact, and clearly $K \subseteq H \subseteq U$.

Section 5

1. Take a sequence { x_n } such that $x_n \in F_n$. For ε >0, we have *k* such that $d_k < \varepsilon$.
   Then for *n,m*>*k*, $x_n, x_m \in F_k$ and *d*(x_n, x_m)< ε .So { x_n }is a Cauchy sequence and hence converges to some $x \in X$. *x* is a a limit point to every F_n and $\bigcap_{n=1}^{\infty} F_n$ is closed, we have $x \in \bigcap_{n=1}^{\infty} F_n$. It is not true if *X* is not complete. We can see that by taking *X*=(0,1), F_n =(0, $\frac{1}{n}$].
2. The map maps the entire surface into the table. It is a contraction map,hence there is a fixed point *x*, the desired point.
3. Take a Cauchy sequence { f_n }in *X*. Then $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in \mathbb{R}$, hence limit exists. Define ,for each *x*, *f*(*x*)= $\lim_{n\to\infty} f_n(x)$. Pick ε >0, choose *k* such that

   $| f_n(x) - f_m(x) | < \frac{\varepsilon}{2}$ for all *x* whenever *m,n*>*k*. For $x \in \mathbb{R}$, choose *p*>*k* so that

   $| f_p(x) - f(x) | < \frac{\varepsilon}{2}$.Then for *n*>*k*, $| f_n(x) - f(x) | \leq | f_n(x) - f_p(x) | + | f_p(x) - f(x) | < \varepsilon$. This shows that $f_n \to f$ in metric *d*.

   It remains to show that *f* is continuous. For $x \in \mathbb{R}$ and ε >0, choose *n* such that $d(f, f_n) < \frac{\varepsilon}{3}$.

   As f_n is continuous at *x*, we can choose δ >0 such that $| f_n(x) - f(x) | < \frac{\varepsilon}{3}$ whenever $y \in (x-\delta, x+\delta)$. Then |*f*(*x*-*f*(*y*)|≤| $f(x) - f_n(x) | + | f_n(x) - f_n(y) | + | f_n(y) - f(y) | < \varepsilon$.
4. By exercise 5 in section 2, if such function exists, then $\bigcap_{q\in\mathbb{Q}} \mathbb{R} \setminus qK$ for some open

   Sets V_n. Since V_n contains $\mathbb{Q}$, V_n is dense. Therefore $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} \mathbb{R} \setminus V_n$ is a countable union of nowhere dense sets. Hence $\mathbb{R} = \left\{ \bigcup_{n=1}^{\infty} \mathbb{R} \setminus V_n \right\} \cup \left\{ \bigcup_{q\in\mathbb{Q}} \{q\} \right\}$ is a countable union of nowhere dense sets, a contradiction to the corollary of Theorem 5.7.
5. (i)At the n^{th} construction of *K* , the remaining intervals have total length less than 2^{-n} . For any interval (*a*,*b*), we have (*a*,*b*)*K* nonempty and open. Hence contains an open interval.
   (ii)*qK* is nowhere dense for all rational *q*. Also, $\bigcup_{q\in\mathbb{Q}} qK$ contains all rationals and hence dense.

If $\bigcup_{q\in\mathbb{Q}} qK$ is closed, then $\bigcup_{q\in\mathbb{Q}} qK = \mathbb{R}$ which is a contradiction to the Baire's Theorem. If $\bigcup_{q\in\mathbb{Q}} qK$ is open, $\mathbb{R} \setminus \bigcup_{q\in\mathbb{Q}} qK = \mathbb{Q}$ is closed.Note that $\mathbb{R} \setminus qK$ is open and dense, then $\bigcap_{q\in\mathbb{Q}} \mathbb{R} \setminus qK$ is dense by Baire's Theorem. Hence $\bigcap_{q\in\mathbb{Q}} \mathbb{R} \setminus qK = \mathbb{R}$, a contradiction.