To see how powerful the Baire's theorem is, we demonstrate an application: we shall show that there exist continuous functions that are nowhere differentiable on $\mathbb{R}$. Indeed, by Baire's theorem, the set of all such functions are dense in the (complete) metric space of all continuous functions, where the metric is induced by the sup norm. We provide a detailed proof below.

Proof. Let C denote the metric space of all continuous functions on $\mathbb{R}$, where the metric is induced by the sup norm. Define, for each g in C and each real number x, y and z, that

$$E_g(x,y,z) = \left| \frac{g(x)-g(y)}{x-y} - \frac{g(x)-g(z)}{x-z} \right|$$

provided that $x \neq y$ and $x \neq z$. For any real numbers a and b with $a < b$, define $A[a, b] \subseteq C$ by

$$A[a,b] = \left\{ g \in C : \forall x \in [a,b],\ \exists y_x, z_x \in [a,b] \text{ such that } E_g(x, y_x, z_x) > 1 \right\}.$$

Our strategy is to show that for any real numbers a and b with $a < b$, we have:

(a) $A[a, b]$ is dense in C; and

(b) $A[a, b]$ is open in C.

From (a) and (b) we conclude, from Baire's Theorem, that

$$\Lambda = \bigcap_{\substack{m\in\mathbb{Z} \\ n\in\mathbb{N}}} A\left[\frac{m}{n}, \frac{m+1}{n}\right]$$

is a (non-empty) dense set in C. (Note we used the completeness of the metric space C here!) Finally we shall prove that any g in Λ is continuous but nowhere differentiable on $\mathbb{R}$. This will complete our proof.

So let us first prove (a):

Given $f \in C$ and $\varepsilon > 0$, we need to construct a function $g \in A[a,b]$ such that $\|g - f\| < \varepsilon$. To give light on what will be going on, let us mention that the g that we will construct will be piecewise linear and of "zig-zag" shape on $[a, b]$, and will satisfy $g = f$ outside (a, b). The way we do this is as follows: By uniform continuity of f on $[a, b]$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2}$$

whenever *x*, *y* are in [*a*, *b*] and $|x-y|<\delta$. Let $p_0, p_1, p_2, \dots, p_n$ be (finitely many) points in [*a*, *b*] such that $a=p_0<p_1<\dots<p_n=b$ and $|p_i-p_{i-1}|<\delta$ for all *i* = 1, 2, … , *n*. Then for all *i* = 1, 2, … , *n*, we have

$$|f(x)-f(p_i)|<\frac{\varepsilon}{2}$$

whenever $x\in[p_{i-1}, p_i]$. Now for each $i\in\{1, 2, \dots, n\}$, we define $q_{i,0}$ as follows: if $f(p_i)\neq f(p_{i-1})$, then we let $q_{i,0}\in(p_{i-1}, p_i)$ be such that

$$\frac{|f(p_i)-f(p_{i-1})|}{q_{i,0}-p_{i-1}}>1;$$

if $f(p_i)=f(p_{i-1})$, then we let $q_{i,0}=p_{i-1}$. Then let k_i be a positive even integer such that

$$\frac{p_i-q_{i,0}}{k_i}<\frac{\varepsilon}{2}$$

and set

$$q_{i,j}=q_{i,0}+j\frac{p_i-q_{i,0}}{k_i}$$

for *i* = 1, 2, …, *n* and *j* = 1, 2, …, k_i, we define a function *g* by setting $g(x)=f(x)$ for all *x* outside (*a*, *b*) and setting *g* to be piecewise linear on [*a*, *b*] such that for *i* = 1, 2, …, *n* and *j* = 0, 1, 2, …, k_i,

$$g(q_{i,j})=\begin{cases} f(p_i) & \text{if } j \text{ is even} \\ f(p_i)+\dfrac{\xi_i\varepsilon}{2} & \text{if } j \text{ is odd} \end{cases},$$

where ξ_i is defined by

$$\xi_i=\begin{cases} -1 & \text{if } f(p_i)>f(p_{i-1}) \\ 1 & \text{if } f(p_i)\le f(p_{i-1}) \end{cases}.$$

Then this *g* is of “zig-zag shape” on [*a*, *b*] (i.e. the slopes of *g* on two adjacent intervals are always of opposite signs), and the slopes on each interval is always of absolute value strictly greater than 1 (i.e. we have

$$\left|\frac{g(q_{i,j})-g(q_{i,j-1})}{q_{i,j}-q_{i,j-1}}\right|=\frac{\varepsilon}{2}\cdot\frac{1}{q_{i,j}-q_{i,j-1}}>1$$

for *i* = 1, 2, …, *n* and *j* = 1, 2, …, k_i, while

$$\left|\frac{g(q_{i,0})-g(p_{i-1})}{q_{i,0}-p_{i-1}}\right|=\frac{|f(p_i)-f(p_{i-1})|}{q_{i,0}-p_{i-1}}>1$$

for those i = 1, 2, …, n for which $f\left(p_i\right) \neq f\left(p_{i-1}\right)$). Hence one can then check that $g \in A[a,b]$ by checking that for each x in $[p_{i-1}, p_i]$, there exists $y_x, z_x \in [p_{i-1}, p_i]$ with $y_x \neq x$ and $z_x \neq x$ such that $E_g(x, y_x, z_x) > 1$. (In fact one can always choose y_x and z_x to be two of the $q_{i,j}$'s.) It is obvious from our construction that $\left|g(x) - f(p_i)\right| < \varepsilon / 2$ for all $x \in [p_{i-1}, p_i]$, so together with and $\left|f(x) - f(p_i)\right| < \varepsilon / 2$ for all $x \in [p_{i-1}, p_i]$ we get $\|g - f\| < \varepsilon$. Thus we have completed the proof of (a).

Now let us prove (b):

Let $g \in A[a,b]$ be given. We want to show that there exists $\delta > 0$ such that whenever $f \in C$ is such that $\|f - g\| < \delta$, we have $f \in A[a,b]$. This is done via establishing two claims:

For our given $g \in A[a,b]$, we first claim that there exists $\varepsilon > 0$ such that for all $x \in [a,b]$, there exists $y_x, z_x \in [a,b] \setminus \{x\}$ such that $E_g(x, y_x, z_x) > 1 + \varepsilon$. The reason is that otherwise we would have, for every positive integer n, that there exists $x_n \in [a,b]$ such that for all $y, z \in [a,b] \setminus \{x_n\}$, we have

$$E_g(x_n, y, z) \leq 1 + \frac{1}{n}.$$

By Bolzano-Weierstrass Theorem, $\left\{x_n\right\}$ has a convergent subsequence, which we still denote by $\left\{x_n\right\}$. Let $\alpha = \lim_{n \to \infty} x_n$. Then $\alpha \in [a,b]$, so there exists $y_\alpha, z_\alpha \in [a,b] \setminus \{\alpha\}$ such that $E_g(\alpha, y_\alpha, z_\alpha) > 1$. But then for all sufficiently large n, we have $x_n \neq y_\alpha$ and $x_n \neq z_\alpha$, and this gives

$$E_g(x_n, y_\alpha, z_\alpha) \leq 1 + \frac{1}{n}$$

for all sufficiently large n. Hence from $\alpha = \lim_{n \to \infty} x_n$ we conclude that

$$1 < E_g(\alpha, y_\alpha, z_\alpha) = \lim_{n \to \infty} E_g(x_n, y_\alpha, z_\alpha) = 1,$$

which is a contradiction. This proves our claim.

Next we claim that for our given $g \in A[a,b]$ and for our $\varepsilon > 0$ as chosen in the previous claim, there exists $r > 0$ such that for every x in [a, b], there exists $y_x, z_x \in [a,b] \setminus (x - r, x + r)$ such that $E_g(x, y_x, z_x) > 1 + \varepsilon$. The reason is that otherwise we would have, for every positive integer n, that there exists $x_n \in [a,b]$

such that there is no $y, z \in [a,b] \setminus (x_n - 1/n, x_n + 1/n)$ which satisfies $E_g(x_n, y, z) > 1 + \varepsilon$. By Bolzano-Weierstrass Theorem again, the sequence $\{x_n\}$ has a convergent subsequence, which we still denote by $\{x_n\}$. Let $\alpha = \lim_{n\to\infty} x_n$. Then $\alpha \in [a,b]$, so there exists $y_\alpha, z_\alpha \in [a,b] \setminus \{\alpha\}$ such that $E_g(\alpha, y_\alpha, z_\alpha) > 1 + \varepsilon$. But then for all sufficiently large *n*, we have $x_n \neq y_\alpha$ and $x_n \neq z_\alpha$, and this gives

$$\lim_{n\to\infty} E_g(x_n, y_\alpha, z_\alpha) = E_g(\alpha, y_\alpha, z_\alpha) > 1 + \varepsilon .$$

Hence for all sufficiently large *n*, we have

$$E_g(x_n, y_\alpha, z_\alpha) > 1 + \varepsilon .$$

Since $\alpha = \lim_{n\to\infty} x_n$, $y_\alpha \neq \alpha$ and $z_\alpha \neq \alpha$, for all sufficiently large *n*, we also have

$$y_\alpha, z_\alpha \in [a,b] \setminus (x_n - 1/n, x_n + 1/n) .$$

This contradicts our choice of $\{x_n\}$, and our second claim is established.

Now we are ready to show, for our given $g \in A[a,b]$, that there exists $\delta > 0$ such that whenever $f \in C$ is such that $\|f - g\| < \delta$, we have $f \in A[a,b]$. In fact we will take

$$\delta = \frac{\varepsilon r}{4},$$

where ε and r are chosen as in the previous two claims. Then for $f \in C$ such that $\|f - g\| < \delta$, if $x \in [a,b]$ is given and $y_x, z_x \in [a,b] \setminus (x - r, x + r)$ are such that $E_g(x, y_x, z_x) > 1 + \varepsilon$, then

$$\begin{aligned}
&\left|E_f(x, y_x, z_x) - E_g(x, y_x, z_x)\right| \\
&= \left|\frac{f(x) - f(y_x) - g(x) + g(y_x)}{x - y_x} - \frac{f(x) - f(z_x) - g(x) + g(z_x)}{x - z_x}\right| \\
&\leq \left|\frac{f(x) - g(x)}{x - y_x}\right| + \left|\frac{f(y_x) - g(y_x)}{x - y_x}\right| + \left|\frac{f(x) - g(x)}{x - z_x}\right| + \left|\frac{f(z_x) - g(z_x)}{x - z_x}\right| \\
&< \frac{\delta}{r} + \frac{\delta}{r} + \frac{\delta}{r} + \frac{\delta}{r} \\
&= \varepsilon
\end{aligned}$$

Hence $E_f(x, y_x, z_x) > 1$, and $f \in A[a,b]$, as desired. This completes the proof of (b).

As explained at the beginning of the proof, combining (a) and (b) and invoking Baire's Theorem on the complete metric space *C*, we conclude that

$$\Lambda = \bigcap_{\substack{m\in\mathbb{Z} \\ n\in\mathbb{N}}} A\left[\frac{m}{n}, \frac{m+1}{n}\right]$$

is non-empty and dense in C. Finally, let us prove that any g in Λ is continuous but nowhere differentiable on $\mathbb{R}$:

It is clear from our construction that any g in Λ is continuous on $\mathbb{R}$. To see any such g is nowhere differentiable on $\mathbb{R}$, let x be an arbitrary real number. We will show that g is not differentiable at x. In fact the reason is simple: it is because for each positive integer n, there exists integer m such that

$$x \in \left[\frac{m}{n}, \frac{m+1}{n}\right].$$

Then since g is in $A\left[\frac{m}{n}, \frac{m+1}{n}\right]$, we conclude that there exists $y_n, z_n \in \left[\frac{m}{n}, \frac{m+1}{n}\right]$ such that

$$\left|\frac{g(x)-g(y_n)}{x-y_n} - \frac{g(x)-g(z_n)}{x-z_n}\right| = E_g(x, y_n, z_n) > 1.$$

Now as n tends to infinity, we have $y_n, z_n \to x$, so g cannot be differentiable at x. Hence g is nowhere differentiable on $\mathbb{R}$, and our proof is complete.