Teaching Module (Iterations, Fractals and Chaos)* Teachers' Guide

Mathematical Database (http://www.mathdb.org)

Target: Form 4-7 students who are interested in mathematics.

In this module we aim to guide students to explore, under a unified framework, some phenomena concerning iterations, fractals and chaos. We emphasize concrete examples, and we believe that it is through working out concrete computations that students can have a better feeling of the mathematics behind these deep subjects. Therefore we have designed a series of student-oriented mathematical experiments throughout this module.

The module consists of a total of 7 lessons. In lessons 1 and 2 we introduce various examples of fractals including the Sierpinski Triangle, the Cantor Middle Thirds and the Snowflake. It will be seen how iterations of simple functions lead to complicated geometric objects like fractals. In lesson 3 the logistic map $f(x) = ax(1-x)$ is introduced as a mathematical model of population growth. After that students are led to explore the iterations of this logistic map, and they will be asked to give an interpretation of the observed phenomenon in terms of population growth. Chaos will be observed in lesson 4 via the exploration of the logistic map, and some common misconceptions about chaos will be clarified. Lesson 5 presents a more theoretical explanation about some phenomena observed when we study the iterations of the logistic map in lesson 3. In particular the concept of stability will be emphasized. Towards the end of lesson 5 we guide students to study the iterates of the real quadratic map $f(x) = x^2 + c$, which is a close relative of the logistic map. Then lesson 6 studies the iterates of the complex quadratic map, where students will meet the famous fractals: the Mandelbrot Set and Julia Sets. Finally we conclude the module with lesson 7, where students are led to explore the Newton's method from an iteration point of view. Fractals again naturally arise when we solve equations iteratively using the complex Newton's method. We hope finally students will appreciate how fractals, iterations and chaos come together naturally as a coherent whole.

There are a few routes through which this module can be presented. Apart from going through lessons 1 to 7 in order, one may consider the following possibilities:

- For those who want to avoid differentiation: Lessons 1, 2, 3, 4 and 6 (in order)
- For those who want to avoid complex numbers: Lessons 1, 2, 3, 4, 5 and 7 (in order; skip the last part of lesson 7)
- For those who are only interested in iterations and chaos: Lessons 3, 4, 5 and 7 (in order)

The following flow chart should clarify the interdependence between the lessons:

*This teaching module is part of the Interface Project "A Prelude to Advanced Mathematics" of the Department of Mathematics (CUHK) supported by the University Grant Committee of Hong Kong.

Lesson 1: Iterations and Fractals
To investigate a typical fractal, the Sierpinski Triangle, so as to introduce some fractal phenomena including self-similarity and fractional dimensions.

Lesson 2: More on Fractals
To investigate two other typical fractals, the Cantor Middle Thirds and the Snowflake, so as to consolidate the concepts introduced in Lesson 1.

Lesson 3: Iterations in Real Life – Population Dynamics
To introduce the logistic map as a model of population growth, and to study iterations of the logistic map.

Lesson 4: Chaos!
To explore phenomena arising from iterations of the logistic map (e.g. the 'period doubling' phenomenon), and to clarify some common misconceptions about chaos.

Lesson 5: Stability of Fixed Points and Cycles
To have a more theoretical study of the stability of fixed points and cycles of the logistic map, and to introduce the real quadratic map.

Lesson 6: Fractals again – the Mandelbrot Set and the Julia Sets
To investigate the famous fractals, the Mandelbrot Set and the Julia Sets, via iterations of the complex quadratic map.

Lesson 7: Iterations at Work – Solving Equations by Newton's Method
To study Newton's method from the viewpoint of iterations, and to see how fractals and iterations are related in Newton's method.

Each lesson comes with a worksheet. The worksheets are the basis of classroom activities. The teacher should prepare one copy of worksheet for each student before each lesson. During each lesson, the teacher is suggested to go through the activities on the worksheets with the students. For example, the teacher might wish to explain to the students what exactly the students are expected to do in each mathematical experiment. Whenever a question or a computation comes up in the worksheet, students should be encouraged to think about it or work through it and then discuss it with other classmates. The teacher may ask students to explain their findings or answers to the class and then give comments. A summary for each lesson is given at the end of each worksheet. A list of equipment(s) needed for each lesson is also included at the beginning of each worksheet.

It should be emphasized that this module has been designed so that the role of the teacher is to inspire instead of to teach. It is highly recommended that the teacher encourages students to explore the phenomena on their own, even at the expense of students making mistakes or going the wrong way at times. The students should be able to learn a lot via interacting with their fellow classmates. Nevertheless, hints and guidance from teachers are considered an important part of the learning process of the students, and should be provided skillfully throughout.

Lesson 1 (Iterations and Fractals)

	Suggested duration
Parts 1 - 2: Construction of the Sierpinski Triangle	45 minutes
Part 3: Self-similarity	10 minutes
Parts 4 - 6: Dimension of the Sierpinski Triangle	30 minutes
Part 7: Hitting the Sierpinski Triangle	5 minutes
	Total : 1.5 hours

Parts 1 and 2 lead students to construct the Sierpinski Triangle on transparencies. We also provide a computer program for generating the Sierpinski Triangle using iterations.

In parts 3 to 6 we guide students to explore various fractal behaviour of the Sierpinski Triangle, including self-similarity and a fractional dimension. In particular, we motivate students to discuss what dimension *should be* (instead of what dimension *is*). This should help students appreciate how mathematical definitions are reasonable. After that, we argue that the dimension of the Sierpinski Triangle cannot be 2, because the (2-dimensional) area of the Sierpinski Triangle is 0. We then guide students to find that $\log 3/\log 2$ is a more reasonable dimension of the Sierpinski Triangle. This illustrates that the Sierpinski Triangle possesses a fractional dimension, and it is emphasized that this is a typical property of fractals in general.

Part 7 should be considered a leisurely exposition of students to current research. We illustrate how waves on the Sierpinski Triangle look like. More precisely, we show the students an animation of the propagation of 'waves' on the Sierpinski Triangle, where by 'waves' we mean solutions to the wave equation on the Sierpinski Triangle. This is a deep problem and has only been resolved in the past few years; in fact to discuss waves on the Sierpinski Triangle we need to be able to 'differentiate' on the Sierpinski Triangle, which is not a trivial matter (because the Sierpinski Triangle possesses too many 'holes', unlike $\mathbb{R}^n$). The interested teacher is referred to the following paper:

Strichartz, R. (1999). Analysis on fractals, *Notices pf the American Mathematical Society*, **46**, 1199-1208.

(But the teacher need NOT read this paper in order to be able to present to the students the materials in part 7; we are only including this possible reference to help the interested teacher find some further reading materials.)

Lesson 2 (More on Fractals)

	Suggested duration
Part 1: Cantor Middle Thirds: Length and Dimension	20 minutes
Part 2: Cantor Middle Thirds: Number of Elements	25 minutes
Part 3: Snowflake	20 minutes
Part 4: Limitations of the Similarity Dimension	15 minutes
Part 5: Fractals in Nature	10 minutes
	Total : 1.5 hours

This lesson uses and consolidates the concepts introduced in lesson 1 by using two other well-known fractals, the Cantor Middle Thirds and the Snowflake. As a warm-up, please briefly go over what has been introduced in lesson 1 with the students before the lesson.

Parts 1 and 2 explore the Cantor Middle Thirds. The interesting parts lie in that it is an uncountably infinite set whose 'length' is zero. The students may be surprised to see that the 'number of elements' of a set and the 'length' of a set turn out to be quite different ways of measuring the size of a set; the 'length' criterion says that the Cantor Middle Thirds is small while the 'number of elements' criterion says that the Cantor Middle Thirds is large. This illustrates part of the reason why we need rigorous mathematics.

Part 3 introduces the Snowflake (which is also known as the von Koch curve). It surprises mathematicians by bounding a finite area which has a perimeter of infinite length. In other words, the Snowflake bounds a finite area while its own length is infinite. This has challenged the imagination of many mathematicians for centuries.

Part 4 illustrates that similarity dimension, which has been the only dimension that we introduced so far, is not adequate if we look at some less self-similar fractals. A program is provided to illustrate how the dimension of some 'random fractals' cannot be computed using the similarity dimension. In fact there are many more general notions of dimensions that are known to mathematicians; one of them being the *Hausdorff dimension*. The interested teacher may look it up in any books under the name *fractal geometry*.

As a round-up of the study of fractals so far, part 5 gives some of the many examples of fractals that arise naturally in nature. To this end Mandelbrot's book *The Fractal Geometry of Nature* is a very good reference book. The interested teacher may take a look at this.

Lesson 3 (Iterations in Real Life – Population Dynamics)

	Suggested duration
Part 1: Modelling Population Growth	15 minutes
Part 2: Iterations of the logistic map ($0 < a < 1$)	25 minutes
Part 2: Iterations of the logistic map ($1 < a < 3$)	25 minutes
Part 2: Iterations of the logistic map ($a > 3$)	25 minutes
	Total : 1.5 hours

Part 1 may serve as a first experience of mathematical modelling for students. They are guided to model population growth with the *logistic map*

$$f(x) = ax(1 - x),$$

where a is a parameter between 0 and 4, and x is the population renormalized and takes a value between 0 and 1. If the population at a certain time is x, then $f(x)$ should be thought of the population after one unit time. The formula should model a population quite well, because when the initial population is small (close to 0), the population after unit time should not be too big (since the number of births will not be large); and when the initial population is large (close to 1), then the population after unit time should be small (since many of the species will die as a result of intense competition for resources).

Part 2 is mainly concerned with the phenomena arising from iterations of the logistic map. This can be thought of as the study of the long term behaviour of the population of a certain species. The phenonmena observed when we iterate the logistic map depend on the value of the parameter a. We guide the students to carry out some hands-on experiments that help them understand how the long term behaviour of iterates of the logistic map depends on the parameter a. In particular, the students should realize that:

1. If $0 < a < 1$, then $f^k(x)$ tends to 0 as k tends to infinity.
2. If $1 < a < 3$, then $f^k(x)$ tends to the non-zero fixed point of the logistic map as k tends to infinity.
3. If a is slightly bigger than 3, then $f^k(x)$ stabilizes into a 2-cycle as k tends to infinity.

(Here f^k denotes the k-th fold iterate of f, and x is any typical point in $(0, 1)$.) We do not wish to exclude students who do not have a very rigorous concept of limits; hence heuristic and intuitive proofs should be accepted when they are asked for. In particular, phenomena 2 and 3 above are only explained graphically, and we leave a more careful treatment of the stability of these fixed points and cycles to lesson 5. The study of the iterations of the logistic map continues in lesson 4.

Lesson 4 (Chaos!)

	Suggested duration
Part 1: More about Iterations	40 minutes
Part 2: Chaos	35 minutes
	Total : 1.25 hours

In this lesson, students are guided to further explore the iterations of the logistic map. The teacher may wish to begin the lesson by familiarising students with their experimental findings in the previous lesson. Part 1 studies the logistic map when a is considerably larger than 3. Attractive 4-cycles and 8-cycles will be observed. This leads naturally to the concept of *period doubling*, which is simply the phenomenon that the order of the attractive cycles doubles when the value of the parameter a increases. (We mean, *a priori*, as the parameter value a increases, the order of the attractive cycle may change in any way they like. It turns out that the order of the attractive cycles doubles each time as a increases and passes through a critical value (like $a = 3$ and $a = 1 + \sqrt{6}$). This is referred to as the phenomenon of period doubling.) Next we illustrate that although we have the period doubling phenomenon, 3-cycles, 5-cycles, etc (i.e. cycles whose order is not a power of 2) are also possible. Students will actually see them from the program that we provide on the webpage. Finally we introduce the concept of the *stability* of a fixed point and a cycle. (By a stable fixed point we mean a fixed point such that if you pick an initial point close to that fixed point, then the successive iterates of the initial point must eventually get close to the fixed point; similarly for unstable fixed points, stable cycles and unstable cycles.) The significance of the stability of a fixed point is interpreted using population dynamics.

Part 2 introduces chaos as the situation where the outcome, though deterministic, depends so sensitively on the initial conditions that it is very hard to predict. The term deterministic here is important; by chaos we do not mean random processes whose outcome is probabilistic (or uncertain). By chaos we only mean those phenomenon in which the outcome changes upon a small change of initial data; weather forecasting is probably one of the most well known examples of chaos. If all the current weather conditions are precisely known, then theoretically we must be able to determine the weather after a month. But usually this is not possible, because our measurment can never be so precise and if we miss to take into account some seemingly minor current data (for instance, the flap of a butterfly's wing in South America), then what we predict to happen in the long run (say after a month) will be very different from the actual situation. We hope this will clarify some common myths about the word chaos, which is often (wrongly) used in a non-mathematical sense.

Lesson 5 (Stability of Fixed Points and Cycles)

	Suggested duration
Part 1: Background Materials about Differentiation	10 minutes
Part 2: Iterations of a Linear Map	20 minutes
Part 3: Stability of Fixed Points and Cycles	20 minutes
Part 4: The Logistic Map	20 minutes
Part 5: The Real Quadratic Map	20 minutes
	Total : 1.5 hours

Techniques of differentiation are used in this lesson and in lesson 7. Therefore we include a short section on differentiation that serves as a review of the techniques of differentiation that we will need. If students have not seen differentiation before, this lesson (as well as lesson 7) should be skipped.

Part 2 studies the stability of the fixed point of a linear map. It shows that a fixed point of a linear map is stable if the slope of the linear map has absolute value (strictly) smaller than 1, and is unstable if the slope of the linear map has absolute value (strictly) greater than 1. This, together with the idea of approximating a (differentiable) function with a tangent, allows us to study the stability of a fixed point of a (differentiable) function in part 3. The above theory is put into use in part 4, which uses our logistic map as an example and allows one to explain more theoretically the phenomena observed in lesson 3. Finally part 5 introduces the real quadratic map

$$f(x) = x^2 + c$$

and studies the stability of its fixed points and 2-cycles. This leads us naturally into the study of the complex quadratic map in lesson 6.

Lesson 6 (Fractals again – the Mandelbrot Set and the Julia Sets)

	Suggested duration
Part 1: The Complex Quadratic Map	25 minutes
Part 2: The Mandelbrot Set	25 minutes
Part 3: The Julia Sets	25 minutes
	Total : 1.25 hours

The lesson begins with a review of the relationship between the arithmetic and the geometry of complex numbers. The complex quadratic map

$$f(z) = z^2 + c$$

is then introduced, and as an analogue to the real quadratic map, the students are asked to compute the iterates of a few points under the complex quadratic map and see how the iterates may stabilize into a fixed point, a 2-cycle, etc.

Part 2 introduces the Mandelbrot Set as the subset of parameter space for which the set of all iterates of the point 0 is bounded. Some simple properties of the Mandelbrot Set are described and verified using simple examples.

Part 3 introduces the Julia Sets. For each parameter c, the Julia Set associated with c is defined to be the boundary of the set of points in the complex plane whose successive iterates under $f(z) = z^2 + c$

form a bounded set. The students are guided to determine the Julia Set associated with $c = 0$, which happens to be the unit circle, and some other more complicated fractal Julia Sets are shown. The famous theorem that

The Julia Set associated with a parameter c is connected if and only if c is in the Mandelbrot Set

is also stated (without proof, of course).

The concepts of boundedness of a set, boundary of a set and connectedness of a set are introduced as needed. Some of them are not rigorously defined. Students who do not know complex numbers in advance are advised to skip this lesson.

This is actually the deepest lesson in the module. There are still a lot of researches going on about the Mandelbrot Set and the Julia Sets. We hope students will have an appreciation of the deepness of mathematics after working through this lesson.

Lesson 7 (Iterations at Work – Solving Equations by Newton's Method)

	Suggested duration
Part 1: Newton's Method	45 minutes
Part 2: Convergence Issues	15 minutes
Part 3: Complex Newton's Method and Fractals	15 minutes
	Total : 1.25 hours

The lesson begins by introducing Newton's method as an application of iterations in mathematics. The iterative formula

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}$$

is used and explained. Then the validity of the Newton's method, as well as the rate of convergence of the Newton's method, are discussed. Finally the basin of attraction in the complex Newton's method is shown, which turns out to be a fractal. This lesson concludes the module by bringing iterations and fractals together.

The techniques of differentiation that are needed in this lesson are contained in part 1 of lesson 5.