

How to draw an ellipse by compass and straightedge construction?



It is obvious that we can easily find the way to draw a lot of figures with straight lines by compass and ruler. However, is it possible for us to draw some figures with curves, like ellipse?

I

First of all, it is undoubtedly that all curves in a drawing by compass and straightedge constructions must be done by a compass, so they can only have constant curvature.

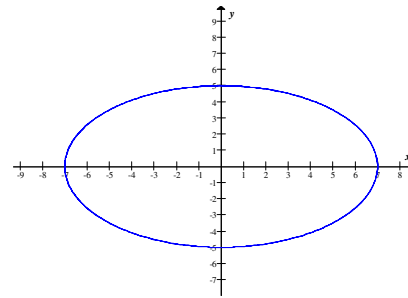
Then we need to know about the curvature equation of an ellipse first.

In this project, I assume that the ellipse has two axis a and b , $a > b$.

$$\text{Let } \mathbf{R}_e(t) = a \cos t \vec{i} + b \sin t \vec{j}$$

$$\mathbf{R}_e'(t) = -a \sin t \vec{i} + b \cos t \vec{j}$$

$$\mathbf{R}_e''(t) = -a \cos t \vec{i} - b \sin t \vec{j}$$



$$\text{curvature } \kappa = \frac{|x' y'' - y' x''|}{\|\mathbf{R}'\|^3}, \quad (\mathbf{x} = \mathbf{x}(t), \mathbf{y} = \mathbf{y}(t), \mathbf{R} = \mathbf{R}_e(t))$$

$$\text{The curvature of } \mathbf{R}_e(t), \kappa_e = \frac{(-a \sin t)(-b \sin t) - (-a \sin t)(b \cos t)}{\| -a \sin t \vec{i} + b \cos t \vec{j} \|^3}$$

$$= \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{\frac{3}{2}}}$$

$$= ab(a^2 \sin^2 t + b^2 \cos^2 t)^{-\frac{3}{2}}$$

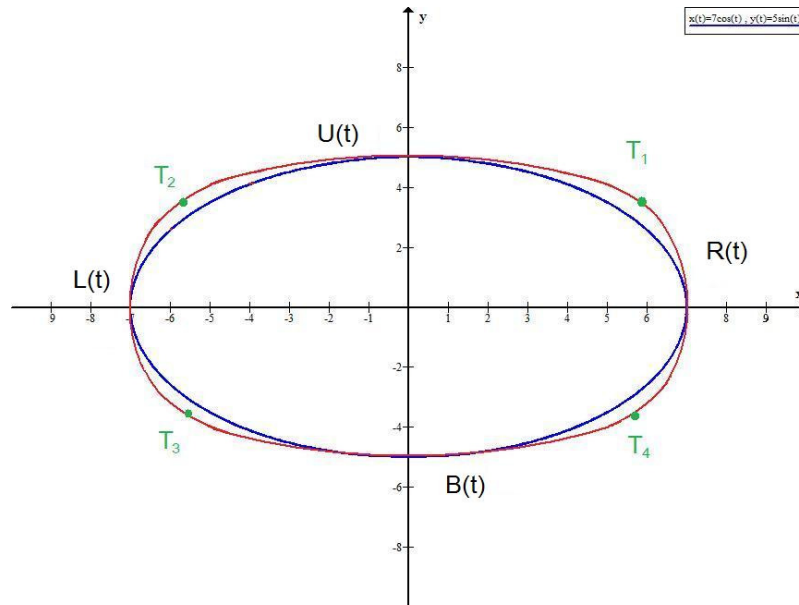
So, the curvature of an ellipse is obviously not a constant unless $a = b$ (which is a circle)

Therefore, drawing an ellipse is impossible to be done by compass and straightedge construction.

II

Now, it is clear that using a compass to draw an ellipse is impossible. However, an ellipse is quite similar to a figure joined by four circle arcs. Can the ellipse be approximated by arcs?

Just like the following example.



Then here comes another question, if I want to approximate the ellipse by arcs, where should the centres of the circles be? Also, how about the length of the radius?

Since the ellipse is symmetric along the x - and y - axis, the centres of the circles should be marked on the x - and y - axis.

III

We know that the circles of the four approximating arcs are tangent to each other. We then let the radius of the two circles be R and r , which $R > r$.

In order to find the way to sketch the approximating arcs, I would like to discuss the relationship between R and r first.

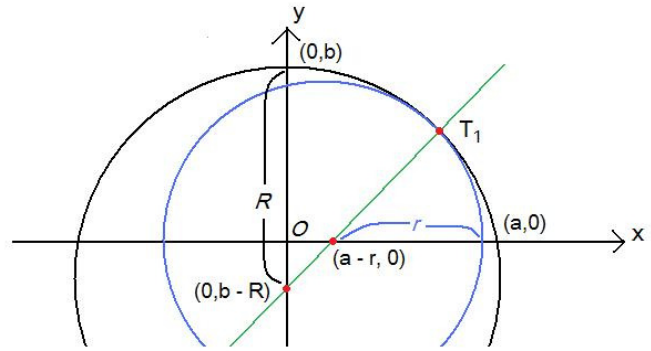
By the property of tangent circles, the tangent point and two centres of circles will be at the same straight line.

Then we can know that there is one and only one value of r corresponding to R , and vice versa.

From the figure, considering the straight line in green, we can obtain an equation:

$$r = R - \sqrt{(b - R)^2 + (a - r)^2}$$

$$\Rightarrow r = \frac{b^2 + a^2 - 2bR}{2(a - R)}$$



IV

We know that the equations of the four arcs (or circles) are:

$$L(t) = [r \cos t - (a - r)]\vec{i} + (r \sin t)\vec{j}$$

$$R(t) = [r \cos t + (a - r)]\vec{i} + (r \sin t)\vec{j}$$

$$U(t) = R \cos t \vec{i} + [R \sin t + (b - R)]\vec{j}$$

$$B(t) = R \cos t \vec{i} + [R \sin t - (b - R)]\vec{j}$$

We can obtain one of the Tangent Points T by solving the simultaneous equations of two equations above.

Then we can find T_1 first. Let it be $x\vec{i} + y\vec{j}$

$$\begin{cases} R(t) = [r \cos t + (a - r)]\vec{i} + (r \sin t)\vec{j} \\ U(t) = \vec{i} + [R \sin t + (b - R)]\vec{j} \end{cases}$$

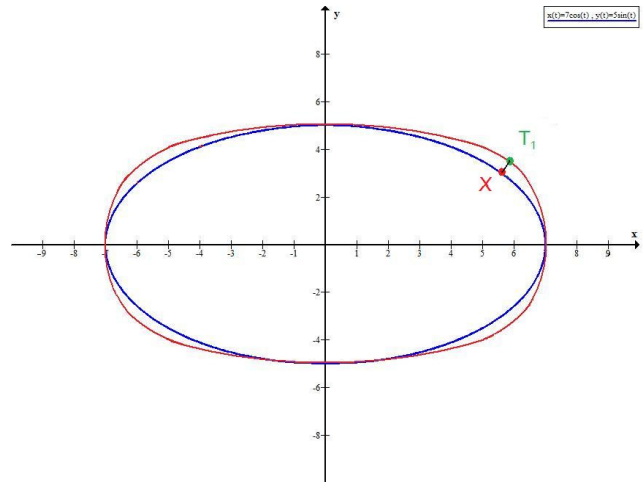
$$\begin{cases} x = r \cos t + (a - r) = R \cos t \\ y = r \sin t = R \sin t + (b - R) \dots (*) \end{cases}$$

$$t = \tan^{-1} \frac{R - b}{a - r}$$

We can then find $x\vec{i}$ and $y\vec{j}$ by substituting t into (*)

V

To make the approximation better, let the minimum distance between t and the ellipse be $|\overline{XT}|$.



Then we can make our approximation better by reducing $|\overline{XT}|$

$$\overline{XT} = (x - a \cos t, y - b \sin t)$$

Let's begin with T_1 .

Let the coordinate of T_1 be (x, y) where x and y can be expressed in terms of R

We know that

$$R_e(t) = (a \cos t, b \sin t)$$

$$R_e'(t) = (-a \sin t, b \cos t)$$

As vector \overline{XT} is orthogonal to R_e' on point X .

$$R_e'(t) \cdot (x - a \cos t, y - b \sin t) = 0$$

$$(-a \sin t)(x - a \cos t) + (b \cos t)(y - b \sin t) = 0$$

$$\frac{a^2 - b^2}{2} \sin 2t + b y \cos t - a x \sin t = 0 \dots (*)$$

Then we can express R in terms of t , which is less complicated

Now, we can find the way to reduce $|\overline{XT}| = \sqrt{(x - a \cos t)^2 + (y - b \sin t)^2}$, which can be

expressed in terms of t .

The minimum value of $|\overline{XT}|$ can be found by solving

$$\begin{cases} \frac{d|\overline{XT}|}{dt} = 0 \\ \frac{d^2|\overline{XT}|}{dt^2} > 0 \end{cases}$$

VI

Let's recall our aim of the task, which is "to approximate an ellipse with arcs drawn by compass and straightedge constructions".

A length that can be drawn by compass and straightedge constructions can be operated by fundamental operations "+", "-", "×", "÷" and " $\sqrt{\quad}$ " only.

Actually, the expression of $|\overline{XT}|$ involves trigonometric function, that means even we can find out the minimum value of $|\overline{XT}|$, the approximation may not be drawn.

However, it can already give us hint that the $|\overline{XT}|$ of our approximation should be around the value we've found. Then we may further find out r and R and finish our drawing.

~ The End ~