From Typhoon to Calculus

During summer of Hong Kong, there are usually a lot of typhoons (tropical cyclones) attacking Hong Kong and the seacoast nearby. Some of them are really violent and dangerous. If we do not do the anti-typhoon actions and go the safety places before the typhoon comes, there will be huge amount of economic loss and even death of people.

The Hong Kong Observatory therefore has to forecast the typhoon, if the typhoon (going to the southeastern China) comes nearest to Hong Kong, the Hong Kong Observatory will hoist the tropical cyclone warning signal.

Therefore, knowing when the typhoon will be most nearest to Hong Kong is quite an important thing.

Finding out when the typhoon comes nearest to Hong Kong, we must first know where the typhoon comes nearest to Hong Kong.

Part I. Where does the typhoon come nearest to Hong Kong?

1. Basic analysis

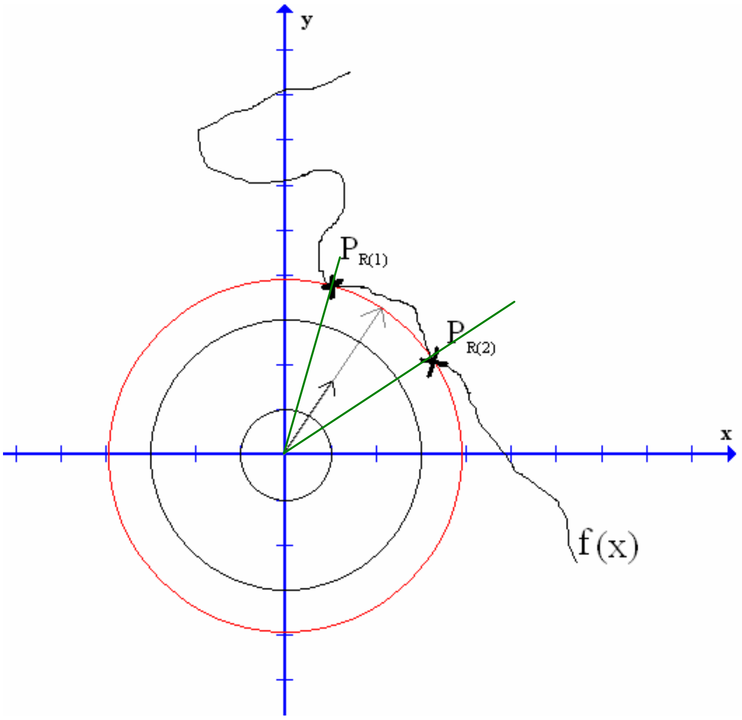
First of all, I would like to put this question into a rectangular coordinate plane. We can let the typhoon be a point and Hong Kong is the origin of the plane. The line of the typhoon's movement can be recorded as a function called $f(x)$. Obviously, this function is a continuous function.

So, I can turn the question "Where does the typhoon come nearest to Hong Kong?" into a mathematical problem: "Finding out the nearest point(s) on $f(x)$ to the origin."

2. What is the condition of "the nearest points"?

To find out the nearest points, I look for the properties that "the nearest points" have first.

I try to imagine that there is a circle with the origin to be its centre; it keeps on enlarging until it touches the function, at points P_R of the set of the nearest points, ***R*** (the set of **R**equiring points).



Similarly, all points in ***R*** are the points of tangency of the graph of *f*(*x*); the normal of these points touches the centre of the circle, the origin.

Therefore, the necessary condition of "the nearest points to the origin" is "***If these points have their normals***[1], these normals touch the origin".

3. Division of the problem

Now, with the aid of the necessary condition above, I can start to find ***R***.

As everybody knows, normal is perpendicular to the tangent. The product of their slopes is -1. Once we find out the tangent, we can find out the normal.

The function of slopes of tangents on *f*(*x*) can be found by differentiation. However, not all functions are totally differentiable. So, I would like to divide this problem into two parts: *f*(*x*) is differentiable or *f*(*x*) is not differentiable.

4. If $f(x)$ is differentiable

The tangent function of point k, $(x_k, f(x_k))$, can be formed as

$y - y_k = m_k (x - x_k)$, m_k means the slope of the tangent of k

By differentiation, $m_k = f'(x_k)$.

As we mentioned above, the product of the slope of tangent (m) and normal (n) is -1.

$\therefore\ n = -\frac{1}{m}$

$\therefore$ The normal function of point k, $(x_k, f(x_k))$ is

$$y - y_k = -\frac{1}{f'(x_k)}(x - x_k)$$

If the normal passes through the origin, that means (0, 0) is one of the solutions of the normal function. Substituting (0, 0) into the normal function, we can obtain an equation.

$$0 - y_k = -\frac{1}{f'(x_k)}(0 - x_k)$$

$$y_k = -\frac{1}{f'(x_k)} x_k \quad \cdot\ \cdot\ \cdot \quad (1)$$

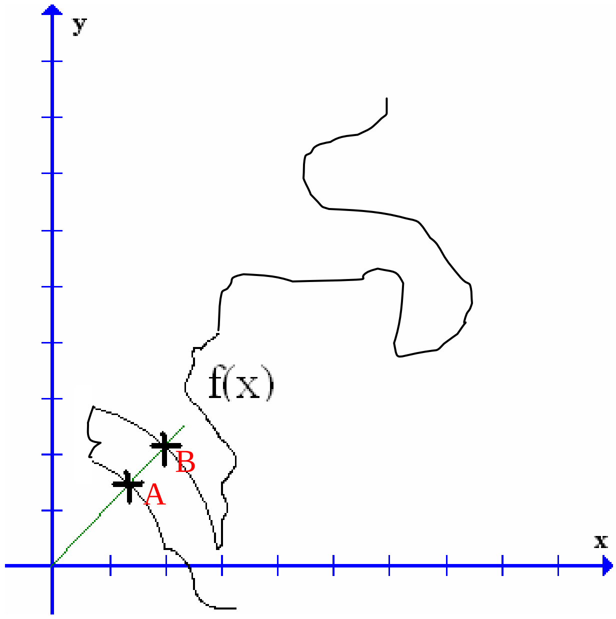
Also, we can all one more equation.

$$y_k = f(x_k) \quad \cdot\ \cdot\ \cdot\ (2)$$

Combining (1) and (2), we will obtain an equation:

$$f(x_k) = -\frac{1}{f'(x_k)} x_k \quad \cdot\ \cdot\ \cdot\ (\ast)$$

Actually, the set of results of this equation, ***K***, may be larger than the set of the nearest points, ***R***. It is because the existence of this kind of functions.



In the graph shown above, the points A and B are both solutions of the normal function and $f(x)$. Obviously, point B $\notin \boldsymbol{R}$. How can we eliminate points like B?

To eliminate points like B, a direct method is to compare all the solutions (x_k, y_k).

$\because$ The distance between the origin and the nearest points k

$= \sqrt{x_k^{\ 2} + y_k^{\ 2}}$

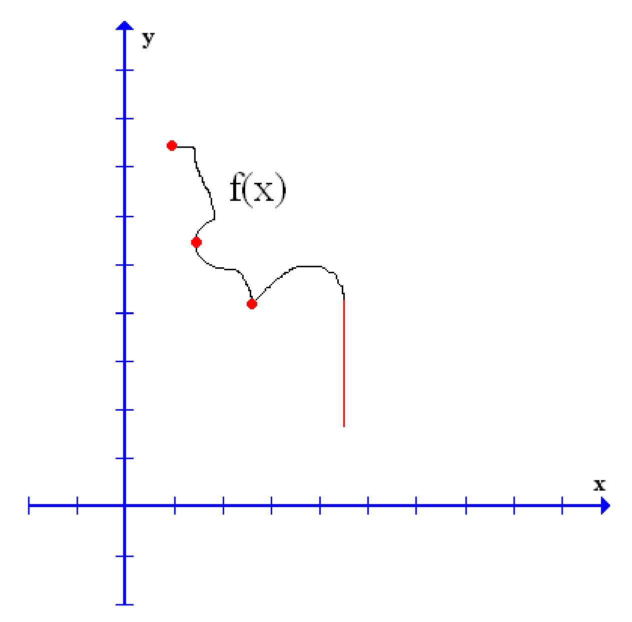
$\therefore$ I have to add one more requirement for $\boldsymbol{R}$:

$$\boldsymbol{R} = \{(x_k, y_k) \mid \min\left(\sqrt{x_k^{\ 2} + y_k^{\ 2}}\right)\},\ x_k, y_k \in \boldsymbol{K}$$

4. What will happen if the function is not completely differentiable?

After using the "General Key"[2] Calculus to solve our problem partly, I move my focus on other parts which are still locked.

It is quite a pity that not all the functions are completely differentiable even though they are continuous (Some continuous functions are even ***totally non-differentiable!***). Just like the following example:



From this graph, we can see some parts of a function is non-differentiable (*the red parts*), including ending points, critical points which their slopes of left and right hand sides are different, points whose tangents are undefined, and also, lines parallel to the y-axis.

Luckily, there are usually not many non-differentiable points in a function.

To solve this problem, we can first find out the set ***K*** of the function. Then check with the rest non-differentiable points.

As to the function of a straight line parallel to the y-axis (completely non-differentiable functions), in fact, they are almost impossible to be found in typhoon as the function has to move strictly parallel to the y-axis.

For this kind of functions, the nearest point of the function to the origin is $(0, e) := \{e \mid \min(|y|)\}$.

5. Conclusion on Part I

After the analysis and calculation, I can finally draw a conclusion on how to find out the set of nearest points, ***R***.

$$\boldsymbol{K} = \{ (x_k, y_k) \mid y_k = f(x_k) = -\frac{1}{f'(x_k)} x_k \}$$

$$\boldsymbol{E} = \{ (x_e, y_e) \mid \text{non-differentiable points of } f(x) \}$$

$$\boldsymbol{R} = \{ (x, y) \mid \min (\sqrt{x_k^2 + y_k^2} \text{ and } \sqrt{x_e^2 + y_e^2}) \}$$

Part II. When does the typhoon come nearest to Hong Kong?

Now I've found out where the typhoon comes nearest to Hong Kong (***R***), it's time to move on to the requiring time of the typhoon needed for travelling to ***R***.

1. Finding out the function of distance travelled-and-time

The Hong Kong Observatory can use their apparatus to measure the speed of the typhoon time to time. By recording the speeds of the typhoon, we can work out a table and then approximate the function of the speed of the typhoon, $g(t)$.

In fact, the function worked out, $g(t)$, is the first-order differential function of the travelled distance-and-time function, $G(t)$. If I can find out $G(t)$ and substitute the distance travelled into $G(t)$, the whole problem is solved.

As $G(t)$ is the anti-differential of $g(t)$, I can use Integral to solve it.

$$G(t) = \int g(t)\, dt + C$$

Since the speed of typhoon must be 0 at the time 0, $G(t)$ passes through the origin. So, we can find out the constant C of $G(t)$ by substituting (0, 0) into the $\int g(t)\, dt + C$.

2. Finding out the distance travelled

This task will be an easy job with the aid of Integral.

By the formula of arc length, the distance of $f(x)$ from P_0 to P_R ($D_{P_0 \to P_R}$) can be calculated by:

$$D_{P_0 \to P_R} = \int_{P_0}^{P_R} \sqrt{1+[f(x)]^2}\,dx, \quad P_0, P_R \in \boldsymbol{R}$$

In fact, $D_{P_0 \to P_R} = G(P_R)$. Therefore, the requiring time t_R can be found out by substituting $\int_{P_0}^{P_R} \sqrt{1+[f(x)]^2}\,dx$ into $G(t)$.

Part III. Conclusion

After the calculation, I can finally draw out my conclusion on the problem:

"When does a typhoon come nearest to Hong Kong?"

Here is the result:

Let $f(x)$ be the route function of the typhoon, $g(t)$ be the function of the speeds of typhoon against the time, and $G(t)$ be the travelled distance of typhoon against the time.

Let $\boldsymbol{R}$ be the set of the nearest points of the typhoon's route to Hong Kong, t_R be the requiring time for the typhoon to go to these points.

First of all, we find out the set of the nearest points to Hong Kong on $f(x)$, ***R***.

$$\boldsymbol{K} = \{ (x_k, y_k) \mid f(x_k) = -\frac{1}{f'(x_k)} \quad x_k = y_k \}$$

$$\boldsymbol{E} = \{ (x_e, y_e) \mid \text{non-differentiable points of } f(x) \}$$

$$\boldsymbol{R} = \{ (x, y) \mid \min (\sqrt{x_k^2 + y_k^2} \text{ and } \sqrt{x_e^2 + y_e^2}) \}$$

Then we obtain $G(t)$ from $g(t)$,

$$G(t) = \int g(t)\, dt + C \,, \quad * \; G(t) \text{ passes though } (0, 0)$$

Finally, we find out the value of $G(t_R)$ and substitute it into G(t) for obtaining t_R.

$$G(t_R) = \int_{P_0}^{P_R} \sqrt{1 + [f(x)]^2}\, dx \,, \quad P_0, P_R \in \boldsymbol{R}$$

Then, we add t_R to the time when the typhoon is on P_0 . So, we can know when the typhoon goes to P_R , the nearest positions to Hong Kong .

Remember, there may not be just one t_R.

~The End~

Remarks: 1. If the points of a function is non-differentiable, it does not have a normal, or its normal cannot be found by differentiation.

2. This term is chosen from this quotation:

*"The method of Fluxions (the calculus) is the **general key** by help where of the modern mathematicians unlock the secrets of Geometry, and consequently of Nature. —— George Berkeley".*